The low match number limit for a barotropic model of radiative flow

Raphaël Danchin, Bernard Ducomet

THE LOW MACH NUMBER LIMIT FOR A BAROTROPIC MODEL OF RADIATIVE FLOW

RAPHAËL DANCHIN*, BERNARD DUCOMET**

*Université Paris-Est, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, Institut Universitaire de France, 61 avenue du Général de Gaulle, 94010 Créteil Cedex 10

**CEA, DAM, DIF, Département de Physique Théorique et Appliquée
F-91297 Arpajon, France

Abstract. We aim at justifying rigorously the low Mach number asymptotics for a model of compressible fluid coupled to the radiation. For simplicity, we restrict ourselves to the barotropic situation, and adopt the so-called $P1$-approximation to model the effects of radiation. We focus on small perturbations of stable constant equilibria. In the critical regularity framework, we establish estimates independent of the Mach number, and convergence results to the incompressible Navier-Stokes equation, exactly as in the non radiative case. Our results hold true in the whole space $\mathbb{R}^n$ as well as in a periodic box $\mathbb{T}^n$ with $n \geq 2$.

Keywords: Radiation hydrodynamics, Navier-Stokes system, low Mach number, critical regularity, $P1$-approximation.

1. Introduction

We consider the barotropic version of a model of radiation hydrodynamics. Our main goal is to provide the rigorous justification of the Low Mach number limit that has been recently investigated formally and numerically by Seaïd et al. [18] [9] [17] in order to simulate fire propagation models in open vehicle tunnels.

The fluid is described by standard classical fluid mechanics for the mass density ϱ and the velocity field $\vec{u}$ as functions of the time $t \in \mathbb{R}_+$ and of the (Eulerian) spatial coordinate x that belongs to the set Ω which is either the whole space $\mathbb{R}^n$ or some periodic box $\mathbb{T}^n$ with $n \geq 2$. Denoting by $\vec{S}_F$ the *radiative momentum source* acting on the fluid, we thus have:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \text{ in } (0,T) \times \Omega, \tag{1.1}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p = \mathrm{div}_x\big(\mu(\nabla_x\vec{u} + {}^t\nabla_x\vec{u})\big) + \nabla_x(\lambda\mathrm{div}_x\vec{u}) - \vec{S}_F \quad \text{in } (0,T)\times\Omega, \tag{1.2}$$

where p stands for the pressure, which is given by $p = P(\varrho)$ (barotropic assumption) for some smooth enough function P. The viscosity coefficients μ and λ are smooth functions of ϱ satisfying

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0.$$

To model radiative effects, we follow the approach of [4]: we introduce a global distribution function, the radiative intensity $\mathcal{I} = \mathcal{I}(t, x, \vec{\omega}, \nu)$, depending on the direction vector $\vec{\omega} \in \mathcal{S}^{n-1}$, where $\mathcal{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$, and on the frequency $\nu \geq 0$. The action of radiation is then expressed in terms of integral means (with respect to the variables

$\vec{\omega}$ and ν) of quantities depending on $\mathcal{I}$. The radiative intensity $\mathcal{I}$ evolves through the following radiative transfer equation:

$$\frac{1}{c}\,\partial_t\mathcal{I} + \vec{\omega}\cdot\nabla_x\mathcal{I} = S \qquad \text{in } (0,T)\times\Omega\times\mathcal{S}^{n-1}\times(0,\infty), \tag{1.3}$$

where c is the speed of light.

The radiative source $S := S_a + S_s$ is the sum of an emission-absorption term:

$$S_{a,e} := \sigma_a\,(B(\nu,\varrho) - \mathcal{I})$$

and of a scattering contribution:

$$S_s := \sigma_s\left(\tilde{\mathcal{I}} - \mathcal{I}\right) \quad \text{where} \quad \tilde{\mathcal{I}} := \frac{1}{|\mathcal{S}^{n-1}|}\int_{\mathcal{S}^{n-1}} \mathcal{I}\, d\vec{\omega}.$$

The transport coefficients $\sigma_a(\varrho,\vec{\omega},\nu)$ and $\sigma_s(\varrho,\vec{\omega},\nu)$ are (given) nonnegative functions. The *distribution function* $B(\nu,\varrho)$ which appears in $S_{a,e}$, measuring the discrepancy from equilibrium, is a barotropic equivalent of the Planck's function, depending smoothly on ϱ.

Finally, the radiative momentum source $\vec{S}_F$ in the r.h.s. of (1.2) is given by

$$\vec{S}_F = \frac{1}{c}\int_0^\infty\int_{\mathcal{S}^{n-1}} \vec{\omega} S\; d\vec{\omega} d\nu.$$

System (1.1), (1.2), (1.3) can be viewed as a simplified model in radiation hydrodynamics [15], [16]. More realistic systems (as regards astrophysics and asymptotic regimes) have been proposed by Lowrie, Morel and Hittinger [14] and revisited recently by Buet and Després [3].

In what follows, we assume that σ_a and σ_s do not depend on $\vec{\omega}$ (isotropy), and make the so-called "gray hypothesis", that is, σ_a and σ_s do not depend on the frequency ν. After integrating with respect to ν, and considering the 'integrated' quantities (still keeping the same notation), this enables us to omit the dependency with respect to ν in (1.1), (1.2), (1.3). Our second simplification is that we consider the so-called "P1-Approximation" of (1.3), that is, we postulate the following decomposition of $I \equiv \int_0^\infty \mathcal{I}\, d\nu$:

$$I = I_0 + \vec{\omega}\vec{I}_1, \tag{1.4}$$

where I_0 and $\vec{I}_1$ do not depend on $\vec{\omega}$ and ν anymore.

This simplification amounts to assume a slight departure of radiation from isotropy [1]. For more complete informations about $P1$ moments method and closure relations, see [3] [2].

Plugging (1.4) into (1.3), taking the first two moments with respect to $\vec{\omega}$ and integrating on $\mathcal{S}^{n-1}\times\mathbb{R}_+$ yields:

$$\frac{1}{c}\,\partial_t I_0 + \frac{1}{n}\,\mathrm{div}_x\vec{I}_1 = \sigma_a(\varrho)(B(\varrho) - I_0) \qquad \text{in } (0,T)\times\Omega, \tag{1.5}$$

$$\frac{1}{c}\,\partial_t\vec{I}_1 + \nabla_x I_0 = (\sigma_a(\varrho) + \sigma_s(\varrho))\vec{I}_1 \qquad \text{in } (0,T)\times\Omega, \tag{1.6}$$

while the radiative source $\vec{S}_F$ rewrites

$$\vec{S}_F = \left(\frac{\sigma_a(\rho) + \sigma_s(\rho)}{n}\right)\vec{I}_1. \tag{1.7}$$

[1] One observes that a P0 approximation, retaining only the isotropic part I_0, would produce a complete decoupling between hydrodynamics and radiative transfer.

In order to identify the appropriate limit regime we perform a general scaling, denoting by $\bar{L}$, $\bar{T}$, $\bar{U}, \bar{\varrho}$, $\bar{p}$, the reference hydrodynamical quantities (length, time, velocity, density, pressure) and by $\bar{I}$, $\bar{\sigma}_a$, $\bar{\sigma}_s$, $\bar{B}$, the reference radiative quantities (radiative intensity, absorption and scattering coefficients and equilibrium function).

Let $Sr := \bar{L}/\bar{T}\bar{U}$, $Ma := \bar{U}/\sqrt{\bar{\varrho}\bar{p}}$ and $Re := \bar{U}\bar{\varrho}\bar{L}/\bar{\mu}$ be the Strouhal, Mach, Reynolds (dimensionless) numbers corresponding to hydrodynamics. Let also define $\mathcal{C} := c/\bar{U}$, $\mathcal{L} := \bar{L}\bar{\sigma}_a$, $\mathcal{L}_s := \bar{\sigma}_s/\bar{\sigma}_a$, various dimensionless numbers corresponding to radiation.

Denoting by $\hat{t}$ and $\hat{x}$ the renormalized time and space variables, setting $\sigma_a = \bar{\sigma}_a\hat{\sigma}_a$, and so on, we perform the change of unknowns:

$$(\varrho, \vec{u}, j_0, \vec{j}_1)(t,x) = \big(\bar{\varrho}\,\hat{\varrho}, \bar{U}\,\hat{\vec{u}}, \bar{I}\,\hat{j}_0, \bar{I}\,\hat{\vec{j}}_1\big)(\bar{T}\hat{t}, \bar{L}\hat{x}).$$

Choosing $\bar{B} = \bar{I}$, omitting the carets and the dependence with respect to x in the differential operators ∇ and div leads to the following *scaled* continuity and momentum equations (keeping in mind (1.7)):

$$Sr\ \partial_t\varrho + \mathrm{div}\,(\varrho\vec{u}) = 0, \tag{1.8}$$

$$Sr\ \partial_t(\varrho\vec{u}) + \mathrm{div}\,(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{Ma^2}\nabla p(\varrho) - \frac{1}{Re}\left(\mathrm{div}\left(\mu(\nabla\vec{u} + {}^t\nabla\vec{u})\right) + \nabla(\lambda\mathrm{div}\,\vec{u})\right) = \mathcal{L}\left(\frac{\sigma_a(\varrho) + \mathcal{L}_s\sigma_s(\varrho)}{n}\right)\vec{I}_1, \tag{1.9}$$

while the rescaled radiative unknowns I_0 and $\vec{I}_1$ satisfy:

$$\frac{Sr}{\mathcal{C}}\,\partial_t I_0 + \frac{1}{n}\,\mathrm{div}\vec{I}_1 = \mathcal{L}\sigma_a(\varrho)\,(B(\varrho) - I_0)\,, \tag{1.10}$$

$$\frac{Sr}{\mathcal{C}}\,\partial_t\vec{I}_1 + \nabla I_0 = -\left(\mathcal{L}\sigma_a(\varrho) + \mathcal{L}\mathcal{L}_s\sigma_s(\varrho)\right)\vec{I}_1. \tag{1.11}$$

In all that follows, we suppose that a moderate amount of radiation is present ($\mathcal{L} = O(1)$) in our strongly under-relativistic flow ($\mathcal{C}^{-1} = o(1)$) and assume that $\bar{\sigma}_a$ and $\bar{\sigma}_s$ are comparable (i.e. $\mathcal{L}_s \approx 1$). To clarify the presentation, we shall focus on the case where

$$Ma = \varepsilon, \quad Sr = Re = 1, \quad \mathcal{C} = \varepsilon^{-1}\widetilde{\mathcal{C}}, \quad \mathcal{L} = 1 \ \text{ and } \ \mathcal{L}_s = 1,$$

where ε is a small positive number and[2] $\widetilde{\mathcal{C}}$ is bounded from below when $\varepsilon \to 0$. Therefore the rescaled unknowns $(\varrho^\varepsilon, \vec{u}^\varepsilon, I_0^\varepsilon, \vec{I}_1^\varepsilon)$ satisfy

$$\partial_t\varrho^\varepsilon + \mathrm{div}\,(\varrho^\varepsilon\vec{u}^\varepsilon) = 0, \tag{1.12}$$

$$\partial_t\left(\varrho^\varepsilon\vec{u}^\varepsilon\right) + \mathrm{div}\,(\varrho^\varepsilon\vec{u}^\varepsilon\otimes\vec{u}^\varepsilon) + \frac{\nabla p^\varepsilon}{\varepsilon^2} = \mathrm{div}\left(\mu^\varepsilon(\nabla\vec{u}^\varepsilon + {}^t\nabla\vec{u}^\varepsilon)\right) + \nabla(\lambda^\varepsilon\mathrm{div}\,\vec{u}^\varepsilon) + \left(\frac{\sigma_a^\varepsilon + \sigma_s^\varepsilon}{n}\right)\vec{I}_1^\varepsilon, \tag{1.13}$$

$$\frac{\varepsilon}{\widetilde{\mathcal{C}}}\,\partial_t I_0^\varepsilon + \frac{1}{n}\,\mathrm{div}\,\vec{I}_1^\varepsilon = \sigma_a^\varepsilon\left(B^\varepsilon - I_0^\varepsilon\right), \tag{1.14}$$

$$\frac{\varepsilon}{\widetilde{\mathcal{C}}}\,\partial_t\vec{I}_1^\varepsilon + \nabla I_0^\varepsilon = -\left(\sigma_a^\varepsilon + \sigma_s^\varepsilon\right)\vec{I}_1^\varepsilon, \tag{1.15}$$

where we denoted $p^\varepsilon = P(\varrho^\varepsilon)$, $\mu^\varepsilon = \mu(\varrho^\varepsilon)$, $\sigma_a^\varepsilon = \sigma_a(\varrho^\varepsilon)$ and so on.

[2]Recall that as $\mathcal{C}Ma = c/\sqrt{\bar{\varrho}\bar{p}}$, the strongly under-relativistic assumption implies that $\widetilde{\mathcal{C}}$ is large.

In order to compute the limit system, we consider the formal expansions

$$\left\{\begin{aligned} I_0^\varepsilon &= I_0^0 + \tfrac{\varepsilon}{\widetilde{\mathcal{C}}} I_0^1 + \mathcal{O}(\varepsilon^2),\\ \vec{I}_1^\varepsilon &= \vec{I}_1^0 + \tfrac{\varepsilon}{\widetilde{\mathcal{C}}} \vec{I}_1^1 + \mathcal{O}(\varepsilon^2),\\ \varrho^\varepsilon &= \varrho_0 + \varepsilon\varrho_1 + \mathcal{O}(\varepsilon^2),\\ \vec{u}^\varepsilon &= \vec{u}_0 + \varepsilon\vec{u}_1 + \mathcal{O}(\varepsilon^2). \end{aligned}\right. \tag{1.16}$$

Identifying the low order terms, we discover that ϱ_0 must be constant. Denoting by $\bar{\varrho}$ that constant, we thus get

$$\operatorname{div}\vec{u}_0 = 0, \tag{1.17}$$

$$\bar{\varrho}\partial_t\vec{u}_0 + \bar{\varrho}\operatorname{div}(\vec{u}_0\otimes\vec{u}_0) + \nabla\Pi = \mu(\bar{\varrho})\Delta\vec{u}_0, \tag{1.18}$$

$$\nabla I_0^0 = -[\sigma_a(\bar{\varrho}) + \sigma_s(\bar{\varrho})]\vec{I}_1^0, \tag{1.19}$$

$$\frac{1}{n}\operatorname{div}\vec{I}_1^0 = \sigma_a(\bar{\varrho})[B(\bar{\varrho}) - I_0^0], \tag{1.20}$$

which is an incompressible Navier-Stokes system decoupled from two stationary transport equations.

We also get two equations for the radiative correctors, namely

$$\frac{1}{\widetilde{\mathcal{C}}}\partial_t I_1^0 + \nabla I_0^1 = -\frac{1}{\widetilde{\mathcal{C}}}[\sigma_a(\bar{\varrho}) + \sigma_s(\bar{\varrho})]\vec{I}_1^1 - [\partial_\varrho\sigma_a(\bar{\varrho}) + \partial_\varrho\sigma_s(\bar{\varrho})]\varrho_1\vec{I}_1^0, \tag{1.21}$$

$$\frac{1}{\widetilde{\mathcal{C}}}\partial_t I_0^0 + \frac{1}{n\widetilde{\mathcal{C}}}\operatorname{div}\vec{I}_1^1 = \varrho_1\partial_\varrho\sigma_a(\bar{\varrho})[B(\bar{\varrho}) - I_0^0] + \sigma_a(\bar{\varrho})[\varrho_1\partial_\varrho B(\bar{\varrho}) - \widetilde{\mathcal{C}}^{-1}I_0^1]. \tag{1.22}$$

The rest of the paper is devoted to proving rigorously the convergence to the incompressible Navier-Stokes equations. In the next section, we reformulate the low Mach number problem, introduce the functional framework we shall work in and state our main result: global well-posedness for small perturbations of a linearly stable constant reference state $(\varrho^\varepsilon, \vec{u}^\varepsilon, I_0^\varepsilon, \vec{I}_1^\varepsilon) = (\bar{\varrho}, \vec{0}, B(\bar{\varrho}), \vec{0})$, and rigorous derivation of the above asymptotics. The proof strongly relies on a fine analysis of the linearized equations about $(\bar{\varrho}, \vec{0}, B(\bar{\varrho}), \vec{0})$, that is performed in Section 3. Next we come to the proof of the global existence result (Section 4) and, finally, to the study of the convergence when the Mach number goes to 0 (Section 5). Some useful estimates pertaining to a toy 2×2 linear system of ODE may be found in Appendix.

2. Results

To simplify the presentation, we shall assume from now on that the viscosity and radiation coefficients are independent of ϱ (for the nonconstant case see Remark 4.1 below). We shall focus on perturbations of some constant reference state

$$(\varrho^\varepsilon, \vec{u}^\varepsilon, I_0^\varepsilon, \vec{I}_1^\varepsilon) = (\bar{\varrho}, \vec{0}, B(\bar{\varrho}), \vec{0}) \quad\text{with}\quad P'(\bar{\varrho}) > 0 \quad\text{and}\quad B'(\bar{\varrho}) > 0.$$

It is thus natural to introduce the new unknowns $j_0^\varepsilon := I_0^\varepsilon - B(\bar{\varrho})$ and $\vec{j}_1^\varepsilon := \vec{I}_1^\varepsilon$. Furthermore, as we expect to have $\varrho^\varepsilon = \bar{\varrho} + \mathcal{O}(\varepsilon)$, and as we prefer to work with linear equations for j_0^ε and $\vec{j}_1^\varepsilon$, we define

$$b^\varepsilon := \frac{1}{\widetilde{\varepsilon}}\left(\frac{B(\varrho^\varepsilon) - B(\bar{\varrho})}{\bar{\varrho}B'(\bar{\varrho})}\right) \quad\text{with}\quad \widetilde{\varepsilon} := \frac{\varepsilon}{\sqrt{P'(\bar{\varrho})}}\cdot$$

Of course, as $B'(\bar\varrho)>0$, the functions depending on ϱ^ε may be expressed in terms of b^ε for small enough perturbations of $\bar\varrho$. The governing equations for $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ thus read

$$\left\{\begin{aligned}
&\partial_t b^\varepsilon+\vec u^\varepsilon\cdot\nabla b^\varepsilon+\frac{\operatorname{div}\vec u^\varepsilon}{\widetilde\varepsilon}=k_1(\widetilde\varepsilon b^\varepsilon)\operatorname{div}\vec u^\varepsilon,\\
&\partial_t\vec u^\varepsilon+\vec u^\varepsilon\cdot\nabla\vec u^\varepsilon-\frac{\mathcal A\vec u^\varepsilon}{\bar\varrho}+\frac{\nabla b^\varepsilon}{\widetilde\varepsilon}-\left(\frac{\sigma_a+\sigma_s}{n\bar\varrho}\right)\vec j_1^\varepsilon=\frac{k_2(\widetilde\varepsilon b^\varepsilon)\mathcal A\vec u^\varepsilon}{\bar\varrho}\\
&\qquad\qquad+\frac{k_3(\widetilde\varepsilon b^\varepsilon)}{\widetilde\varepsilon}\,\nabla b^\varepsilon+\left(\frac{\sigma_a+\sigma_s}{n\bar\varrho}\right)k_4(\widetilde\varepsilon b^\varepsilon)\vec j_1^\varepsilon,\\
&\tfrac{\varepsilon}{\widetilde{\mathcal C}}\,\partial_t j_0^\varepsilon+\tfrac1n\,\operatorname{div}\vec j_1^\varepsilon=\sigma_a\left(\bar\varrho B'(\bar\varrho)\widetilde\varepsilon b^\varepsilon-j_0^\varepsilon\right),\\
&\tfrac{\varepsilon}{\widetilde{\mathcal C}}\,\partial_t\vec j_1^\varepsilon+\nabla j_0^\varepsilon=-\left(\sigma_a+\sigma_s\right)\vec j_1^\varepsilon,
\end{aligned}\right.\tag{2.1}$$

with $\mathcal A:=\mu\Delta+(\lambda+\mu)\nabla\operatorname{div}$ and where k_1, k_2, k_3 and k_4 are smooth functions vanishing at 0.

Before stating our main result, let us specify the functional framework we shall work in. Roughly, we shall adopt the same critical regularity framework as in our first paper [7]. However, we will have to set norms depending on the parameters ε and $\widetilde{\mathcal C}$ in order to get optimal estimates, enabling us to study the low Mach number asymptotics.

Let us first very briefly recall the definition of homogeneous Besov spaces $\dot B^s_{2,1}$ (the reader is referred to [1], Chap. 2 for more details). For simplicity, we focus on the $\mathbb R^n$ case, the adaptation to the periodic setting being quite strandard. Fix some smooth radial bump function $\chi:\mathbb R^n\to[0,1]$ with $\chi\equiv1$ on $B(0,1/2)$ and $\chi\equiv0$ outside $B(0,1)$, nonincreasing with respect to the radial variable. Let $\varphi(\xi):=\chi(\xi/2)-\chi(\xi)$. The elementary spectral cut-off operator entering in the Littlewood-Paley decomposition is defined by

$$\dot\Delta_j u:=\varphi(2^{-j}D)u=\mathcal F^{-1}(\varphi(2^{-j}D)\mathcal F u),\qquad j\in\mathbb Z$$

where we denote by $\mathcal F$ the standard Fourier transform in $\mathbb R^n$.

For any $s\in\mathbb R$, the *homogeneous Besov space* $\dot B^s_{2,1}$ is the set of tempered distributions u so that

$$\|u\|_{\dot B^s_{2,1}}:=\sum_{j\in\mathbb Z}2^{js}\|\dot\Delta_j\|_{L^2}<\infty$$

and

$$\lim_{\lambda\to+\infty}\chi(\lambda D)u=0\quad\text{in}\quad L^\infty.\tag{2.2}$$

As pointed out in [7], scaling considerations (that neglect low order terms of System (2.1)) suggest that critical regularity is $\dot B^{\frac n2-1}_{2,1}$ for $\vec u_0$, $j_{0,0}$ and $\vec j_{1,0}$, and $\dot B^{\frac n2}_{2,1}$ for b_0. However, being 'out of scaling' the lower order terms may hinder the proof of global-in-time estimates. To overcome this, one has to make additional assumptions for the low frequencies of some unknowns. This motivates our introducing norms where the specific behavior of low frequencies is taken into account. More precisely, for any distribution u satisfying (2.2) and any positive parameter η, we set:

$$\|u\|^{\ell,\eta}_{\dot B^s_{2,1}}:=\sum_{2^k\leq2\eta}2^{ks}\|\dot\Delta_k u\|_{L^2}\quad\text{and}\quad\|u\|^{h,\eta}_{\dot B^s_{2,1}}:=\sum_{2^k\geq\eta/2}2^{ks}\|\dot\Delta_k u\|_{L^2},$$

and also

$$u^{\ell,\eta}:=\sum_{2^k\leq\eta}\dot\Delta_k u\quad\text{and}\quad u^{h,\eta}:=\sum_{2^k>\eta}\dot\Delta_k u.$$

Note that $\|u^{\ell,\eta}\|_{\dot B^s_{2,1}}\leq C\|u\|^{\ell,\eta}_{\dot B^s_{2,1}}$ and $\|u^{h,\eta}\|_{\dot B^s_{2,1}}\leq C\|u\|^{h,\eta}_{\dot B^s_{2,1}}$. Because the Littlewood-Paley decomposition is not quite orthogonal, it is important to allow for a small overlap in the above definition of norms.

From our investigation of the linearized equations in the next section, we shall find out that the 'natural' threshold between low and high frequencies for the radiative unknowns $(j_0,\vec{j}_1)$ is at

$$\widetilde\rho_\varepsilon:=\rho_0\sqrt{\nu\,\widetilde{\mathcal C}/\varepsilon}\quad\text{with}\quad \nu:=\lambda+2\mu, \tag{2.3}$$

where ρ_0 depends only on $n,\ \bar\varrho,\ P'(\bar\varrho),\ B'(\bar\varrho),\ \sigma_a$ and σ_s, and that the threshold between low and high frequencies for b is at $1/(\varepsilon\nu)$, exactly as in the nonradiative case (see [5]).

One can now state our main result.

Theorem 2.1. *Let $\nu_0>0$ and $\widetilde\rho_\varepsilon$ be given by (2.3). Assume that*

$$1\lesssim\widetilde{\mathcal C}\lesssim\nu\varepsilon^{-1}. \tag{2.4}$$

There exist three constants ε_0, c and C depending only on $\bar\varrho$, P, B, σ_a, σ_s, n, λ/μ and ν_0 such that if $0<\varepsilon\leq\varepsilon_0$, $0<\nu\leq\nu_0$ and the data satisfy

$$I_0^\varepsilon:=\|\vec u_0^\varepsilon\|_{\dot B^{\frac n2-1}_{2,1}}+\widetilde{\mathcal C}^{-\frac12}\big(\nu\|(j^\varepsilon_{0,0},\vec j^\varepsilon_{1,0})\|^{\ell,\widetilde\rho_\varepsilon}_{\dot B^{\frac n2-1}_{2,1}}+\|(j^\varepsilon_{0,0},\vec j^\varepsilon_{1,0})\|^{h,\widetilde\rho_\varepsilon}_{\dot B^{\frac n2-1}_{2,1}}\big) +\|b_0^\varepsilon\|^{\ell,\frac{1}{\varepsilon\nu}}_{\dot B^{\frac n2-1}_{2,1}}+\varepsilon\nu\|b_0^\varepsilon\|^{h,\frac{1}{\varepsilon\nu}}_{\dot B^{\frac n2}_{2,1}}\leq c\nu, \tag{2.5}$$

then System (2.1) has a unique global solution $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ with:

- *$b^\varepsilon\in\mathcal C(\mathbb R_+;\dot B^{\frac n2-1}_{2,1}\cap\dot B^{\frac n2}_{2,1})$, $(b^\varepsilon)^{\ell,1/(\varepsilon\nu)}\in L^1(\mathbb R_+;\dot B^{\frac n2+1}_{2,1})$, $(b^\varepsilon)^{h,1/(\varepsilon\nu)}\in L^1(\mathbb R_+;\dot B^{\frac n2}_{2,1})$;*
- *$\vec u^\varepsilon\in\mathcal C(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})\cap L^1(\mathbb R_+;\dot B^{\frac n2+1}_{2,1})$;*
- *j_0^ε and $\vec j_1^\varepsilon$ in $\mathcal C(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})$ and, besides $(\mathrm j_0^\varepsilon)^{\ell,\widetilde\rho_\varepsilon}$ and $(\vec{\mathrm j}_1^\varepsilon)^{h,\widetilde\rho_\varepsilon}$ in $L^1(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})$ with*

$$\mathrm j_0^\varepsilon:=j_0^\varepsilon-c_1\varepsilon b^\varepsilon-c_2\widetilde{\mathcal C}^{-1}\varepsilon\,\mathrm{div}\,\vec u^\varepsilon-c_3\widetilde{\mathcal C}^{-2}\varepsilon^2\mathrm{div}\,\vec j_1^\varepsilon\quad\textit{and}\quad\vec{\mathrm j}_1^\varepsilon:=\vec j_1^\varepsilon-c_4\varepsilon\nabla b^\varepsilon, \tag{2.6}$$

  where the coefficients c_1, c_2, c_3 c_4 may be computed in terms of σ_a, σ_s, $\bar\varrho$, $P'(\bar\varrho)$, $B'(\bar\varrho)$ and n.

Moreover, the following inequality is fulfilled:

$$\begin{aligned}\|\vec u^\varepsilon\|_{L^\infty(\dot B^{\frac n2-1}_{2,1})}+\widetilde{\mathcal C}^{-\frac12}\big(\nu\|(j^\varepsilon_0,\vec j^\varepsilon_1)\|^{\ell,\widetilde\rho_\varepsilon}_{L^\infty(\dot B^{\frac n2-1}_{2,1})}+\|(j^\varepsilon_0,\vec j^\varepsilon_1)\|^{h,\widetilde\rho_\varepsilon}_{L^\infty(\dot B^{\frac n2-1}_{2,1})}\big)\\
+\|b^\varepsilon\|^{\ell,\frac1{\varepsilon\nu}}_{L^\infty(\dot B^{\frac n2-1}_{2,1})}+\varepsilon\nu\|b^\varepsilon\|^{h,\frac1{\varepsilon\nu}}_{L^\infty(\dot B^{\frac n2}_{2,1})}+\nu\|\vec u^\varepsilon\|_{L^1(\dot B^{\frac n2+1}_{2,1})}+\nu\|b^\varepsilon\|^{\ell,\frac1{\varepsilon\nu}}_{L^1(\dot B^{\frac n2+1}_{2,1})}+\frac1\varepsilon\|b^\varepsilon\|^{h,\frac1{\varepsilon\nu}}_{L^1(\dot B^{\frac n2}_{2,1})}\\
+\frac{\nu\widetilde{\mathcal C}^{1/2}}{\varepsilon}\|(\mathrm j^\varepsilon_0,\vec{\mathrm j}^\varepsilon_1)\|^{\ell,\widetilde\rho_\varepsilon}_{L^1(\dot B^{\frac n2-1}_{2,1})}+\frac{\widetilde{\mathcal C}^{1/2}}{\varepsilon}\|(j^\varepsilon_0,\vec j^\varepsilon_1)\|^{h,\widetilde\rho_\varepsilon}_{L^1(\dot B^{\frac n2-1}_{2,1})}\Big)\leq CI_0^\varepsilon.\end{aligned} \tag{2.7}$$

Finally, if the family of data $(b_0^\varepsilon,\vec u_0^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$ satisfies (2.5) and the divergence free part $\mathcal P\vec u_0^\varepsilon$ of $\vec u_0^\varepsilon$ converges to $\vec v_0$ in $\dot B^{\frac n2-1}_{2,1}$ weak $*$, then $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ converges to $(0,\vec v,0,\vec 0)$ in $L^\infty(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})$ weak $*$, where $\vec v\in\mathcal C(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})\cap L^1(\mathbb R_+;\dot B^{\frac n2+1}_{2,1})$ stands for the unique global solution of the incompressible Navier-Stokes equations (1.17)–(1.18), supplemented with initial velocity $\vec v_0$.

A few remarks are in order:

(1) Note that (2.4) corresponds to $\varepsilon^{-1} \lesssim \mathcal{C} \lesssim \nu\varepsilon^{-2}$. We could treat smaller values of $\mathcal{C}$. However, as pointed out in the introduction, this would be unphysical.
(2) The decay estimate for $\vec{j}_1^\varepsilon$ is the key to handling the nonlinear term $k_4(\widetilde{\varepsilon} b^\varepsilon)\vec{j}_1^\varepsilon$ in the velocity equation of (2.1).
(3) More accurate convergence results are available (see Theorems 5.1 and 5.2 below).
(4) We have a similar statement if the coefficients σ_a, σ_s, λ and μ depend *smoothly* on ρ (see Remark 4.1).
(5) We expect *local* results for *large data*, in the spirit of [5]. In particular, the lifespan of the limit system (1.17)–(1.18) should be a lower bound of the lifespan of the corresponding solution to (2.1), for small enough ε. In order to avoid wild computations however, we preferred to concentrate on the small data case.

We end this section with a short description of the method leading to the above statement. Proving the global a priori estimate (2.7) is the main step. Recall that a rougher inequality has been established in our recent paper [7]. Unfortunately, we did not keep track of the physical coefficients of the system therein. As specifying the dependency with respect to ε is fundamental in the study of the low Mach number asymptotics, we will have to refine our previous analysis. In fact, as in the nonradiative case, it is convenient to perform a rescaling so as to avoid terms of order ε^{-1} in the system. This naturally leads to the following change of unknowns:

$$
\begin{aligned}
&b(t,x) := \widetilde{\varepsilon} b^\varepsilon(\widetilde{\varepsilon}^2 t, \widetilde{\varepsilon} x), && \vec{u}(t,x) := \widetilde{\varepsilon}\vec{u}^\varepsilon(\widetilde{\varepsilon}^2 t, \widetilde{\varepsilon} x),\\
&j_0(t,x) := \widetilde{\varepsilon}\,\zeta_0\, j_0^\varepsilon(\widetilde{\varepsilon}^2 t, \widetilde{\varepsilon} x), && \vec{j}_1(t,x) := \widetilde{\varepsilon}\tfrac{\zeta_0}{\sqrt{n}}\,\vec{j}_1^\varepsilon(\widetilde{\varepsilon}^2 t, \widetilde{\varepsilon} x),
\end{aligned}
\tag{2.8}
$$

with $\widetilde{\varepsilon} := (P'(\bar{\varrho}))^{-1/2}\varepsilon$ and

$$
\zeta_0 := \frac{1}{\bar{\varrho}}\frac{1}{\sqrt{\widetilde{\mathcal{C}}B'(\bar{\varrho})}}\left(\frac{P'(\bar{\varrho})}{n}\right)^{\frac14}\sqrt{\frac{\sigma_a+\sigma_s}{\sigma_a}}\,\cdot
$$

We eventually get

$$
\begin{cases}
\partial_t b + \vec{u}\cdot\nabla b + \mathrm{div}\,\vec{u} = k_1(b)\mathrm{div}\,\vec{u},\\
\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} - \nu\widetilde{\mathcal{A}}\vec{u} + \nabla b - \widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2\vec{j}_1 = \nu k_2(b)\widetilde{\mathcal{A}}\vec{u} + k_3(b)\nabla b + \widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2 k_4(b)\vec{j}_1,\\
\partial_t j_0 + \widetilde{\alpha}\widetilde{\mathcal{C}}\mathrm{div}\,\vec{j}_1 + \widetilde{\beta}\widetilde{\mathcal{C}}\varepsilon j_0 - \widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2 b = 0,\\
\partial_t\vec{j}_1 + \widetilde{\alpha}\widetilde{\mathcal{C}}\nabla j_0 + \widetilde{\gamma}\widetilde{\mathcal{C}}\varepsilon\vec{j}_1 = 0,
\end{cases}
\tag{2.9}
$$

with $\nu := \frac{\lambda+2\mu}{\bar{\varrho}}$, $\widetilde{\mathcal{A}} := \mathcal{A}/\nu$,

$$
\begin{aligned}
\widetilde{\alpha} &:= \frac{1}{\sqrt{nP'(\bar{\varrho})}}, & \widetilde{\beta} &:= \frac{\sigma_a}{P'(\bar{\varrho})},\\
\widetilde{\gamma} &:= \frac{\sigma_a+\sigma_s}{P'(\bar{\varrho})}, & \widetilde{\varsigma} &:= \frac{\sqrt{\sigma_a(\sigma_a+\sigma_s)B'(\bar{\varrho})}}{n^{1/4}(P'(\bar{\varrho}))^{5/4}}.
\end{aligned}
$$

According to [7], the linear stability condition for (2.9) reads

$$
\widetilde{\mathcal{C}}\widetilde{\beta}^2\widetilde{\gamma}^2\nu > \widetilde{\alpha}\widetilde{\varsigma}^2(\widetilde{\beta}+\widetilde{\gamma})\varepsilon,
$$

and is thus fulfilled for ε going to 0. Hence System (2.9) is globally well posed for small enough data and ε. However, [7] does not specify the dependency of the smallness condition with respect to ε and $\widetilde{\mathcal{C}}$. In order to prove that it is harmless, we will have to make a very careful analysis of the linearized equations associated to (2.9). In Fourier variables (with respect to $x \in \mathbb{R}^n$), those equations may be seen as a linear system of ODEs with constant coefficients depending on the frequency size ρ. The overall linear system is of mixed type : second order parabolic, first order hyperbolic and zero order dissipative, which complicates the task of finding optimal estimates (see the next section). However, once this system is completely understood, proving (2.7) is rather easy : it is only a matter of paralinearizing the system (to keep the convection terms under control) and use appropriate nonlinear estimates.

Next, Inequality (2.7) combined with weak compactness arguments ensures that there exists $\varepsilon_n \to 0$ so that $(b^{\varepsilon_n}, \vec{u}^{\varepsilon_n}, j_0^{\varepsilon_n}, \vec{j}_1^{\varepsilon_n}) \rightharpoonup (b, \vec{u}, j_0, \vec{j}_1)$. That b, j_0 and $\vec{j}_1$ are zero, and that $\vec{u}$ is divergence free may be seen directly by passing to the limit in (2.1). Proving additional uniform estimates for the time derivative of the incompressible part of $\vec{u}^\varepsilon$ combined with Ascoli theorem (Aubin-Lions type argument) allows to pass to the limit in the velocity equation. We eventually find out that $\vec{v}$ satisfies the incompressible Navier-Stokes equations.

Finally, if $\Omega = \mathbb{R}^n$ then, as in the nonradiative case [5], one may take advantage of the dispersive properties of the acoustic wave equation, to upgrade the weak convergence to strong convergence.

3. Linear analysis

This section is devoted to the study of the linearization of (2.9) about $(b, \vec{u}, j_0, \vec{j}_1) = (0, \vec{0}, 0, \vec{0})$, namely,

$$\left\{\begin{aligned} &\partial_t b + \operatorname{div} \vec{u} = f,\\ &\partial_t \vec{u} - \nu \widetilde{\mathcal{A}} \vec{u} + \nabla b - \varsigma \vec{j}_1 = \vec{g},\\ &\partial_t j_0 + \alpha \operatorname{div} \vec{j}_1 + \beta j_0 - \eta b = 0,\\ &\partial_t \vec{j}_1 + \alpha \nabla j_0 + \gamma \vec{j}_1 = \vec{0}, \end{aligned}\right. \tag{3.1}$$

with

$$\alpha := \widetilde{\alpha}\widetilde{\mathcal{C}}, \quad \beta := \widetilde{\beta}\widetilde{\mathcal{C}}\varepsilon, \quad \gamma := \widetilde{\gamma}\widetilde{\mathcal{C}}\varepsilon, \quad \eta = \varsigma := \widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2. \tag{3.2}$$

The study of the evolution of the divergence free parts $\mathcal{P}\vec{u}$ and $\mathcal{P}\vec{j}_1$ of $\vec{u}$ and $\vec{j}_1$ is obvious as we just have

$$\partial_t \mathcal{P}\vec{u} - \mu \Delta \mathcal{P}\vec{u} = \varsigma \mathcal{P}\vec{j}_1 + \mathcal{P}\vec{g} \quad \text{and} \quad \partial_t \mathcal{P}\vec{j}_1 + \gamma \mathcal{P}\vec{j}_1 = \vec{0}. \tag{3.3}$$

So, as in [7], we focus on the linearized system fulfilled by b, j_0 and the potential parts of $\vec{u}$ and of $\vec{j}_1$. To work with scalar unknowns, we set $d := \Lambda^{-1}\operatorname{div} \vec{u}$ and $j_1 := \Lambda^{-1}\operatorname{div} \vec{j}_1$ (with $\Lambda^s := (-\Delta)^{\frac{s}{2}}$). We eventually get the following system (if $f = 0$ and $\vec{g} = \vec{0}$):

$$\left\{\begin{aligned} &\partial_t b + \Lambda d = 0,\\ &\partial_t d - \Lambda b - \nu \Delta d - \varsigma j_1 = 0,\\ &\partial_t j_0 + \beta j_0 + \alpha \Lambda j_1 - \eta b = 0,\\ &\partial_t j_1 + \gamma j_1 - \alpha \Lambda j_0 = 0. \end{aligned}\right. \tag{3.4}$$

Denoting $\rho := |\xi|$, in Fourier variables, the above system rewrites:

$$\frac{d}{dt}\begin{pmatrix} \widehat{b} \\ \widehat{d} \\ \widehat{j}_0 \\ \widehat{j}_1 \end{pmatrix} + \begin{pmatrix} 0 & \rho & 0 & 0 \\ -\rho & \nu\rho^2 & 0 & -\varsigma \\ -\eta & 0 & \beta & \alpha\rho \\ 0 & 0 & -\alpha\rho & \gamma \end{pmatrix}\begin{pmatrix} \widehat{b} \\ \widehat{d} \\ \widehat{j}_0 \\ \widehat{j}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.5}$$

3.1. Estimates for small ρ.

Making the change of unknown

$$\widehat{U} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\varsigma}{\gamma} \\ -\frac{\eta}{\beta} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \widehat{b} \\ \widehat{d} \\ \widehat{j}_0 \\ \widehat{j}_1 \end{pmatrix}, \tag{3.6}$$

and setting $\alpha' := \alpha + \frac{\varsigma\eta}{\beta\gamma}$, we observe that $\widehat{U} = \widehat{U}(t,\rho)$ satisfies

$$\partial_t \widehat{U} + A_0\widehat{U} + \rho\,(A_1 + B_1)\,\widehat{U} + \nu\rho^2 A_2\widehat{U} = 0, \tag{E}$$

with

$$A_0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 - \frac{\alpha\varsigma\eta}{\beta\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha' \\ 0 & 0 & -\alpha & 0 \end{pmatrix},$$

$$B_1 := -\begin{pmatrix} 0 & 0 & 0 & \frac{\varsigma}{\gamma} \\ 0 & 0 & \frac{\alpha\varsigma}{\gamma} & 0 \\ 0 & \frac{\eta}{\beta} & 0 & 0 \\ \frac{\alpha\eta}{\beta} & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{\varsigma}{\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that both A_0 and A_2 are diagonal (up to a small coefficient as regards A_2) and that the diagonal is nonnegative, but degenerate. As for matrix A_1, it is antisymmetric and still tractable by adapting the analysis of the linearized barotropic Navier-Stokes equations. In fact the main difficulty comes from matrix B_1.

We claim that the change of unknown $\widehat{V} := (I + \rho P)\widehat{U}$ for some suitable matrix P will enable us to annihilate the bad term $\rho B_1\widehat{U}$ if ρ is small enough. Indeed, we observe that

$$\partial_t\widehat{V} + (I+\rho P)A_0(I+\rho P)^{-1}\widehat{V} + \rho(I+\rho P)(A_1+B_1)(I+\rho P)^{-1}\widehat{V} \\ + \nu\rho^2(I+\rho P)A_2(I+\rho P)^{-1}\widehat{V} = 0.$$

Because

$$\begin{aligned}(I+\rho P)^{-1} &= I - \rho P(I+\rho P)^{-1} \\ &= I - \rho P + \rho^2 P^2 (I+\rho P)^{-1} \\ &= I - \rho P + \rho^2 P^2 - \rho^3 P^3 (I+\rho P)^{-1},\end{aligned}$$

we discover that

$$\partial_t\widehat{V} + A_0\widehat{V} + \rho\big(A_1 + B_1 + [P, A_0]\big)\widehat{V} + \rho^2\big([A_0, P]P + [P, A_1] + [P, B_1] + \nu A_2\big)\widehat{V} \\ + \rho^3 (I+\rho P)\big((A_1+B_1)P^2 - A_0P^3 - \nu A_2 P\big)(I+\rho P)^{-1}\widehat{V} = 0.$$

Therefore, if one can choose P so that

$$[A_0, P] = B_1, \tag{3.7}$$

then we end up with

$$\partial_t\widehat{V}+A_0\widehat{V}+\rho A_1\widehat{V}+\rho^2\left(\nu A_2+PB_1+[P,A_1]\right)\widehat{V}=\rho^3(I+\rho P)\,A_3\,(I+\rho P)^{-1}\widehat{V},\tag{3.8}$$

where $A_3:=(PA_0-A_1)P^2+\nu A_2P$. Note that the matrix B_1 now appears in the second order term instead of a first order term before the change of unknown.

In order to determine P, let us rewrite A_0, B_1 and P in block form:

$$A_0=\begin{pmatrix}0&0\\0&D\end{pmatrix},\quad B_1=\begin{pmatrix}0&B_1^1\\B_1^2&0\end{pmatrix},\quad P=\begin{pmatrix}P^{11}&P^{12}\\P^{21}&P^{22}\end{pmatrix}.$$

Computing the commutator

$$[A_0,P]=\begin{pmatrix}0&-P^{12}D\\DP^{21}&[D,P^{22}]\end{pmatrix},\tag{3.9}$$

we see that a convenient choice for P is

$$P^{11}:=0,\quad P^{22}:=0,\quad P^{12}:=-B_1^1D^{-1},\quad P^{21}:=D^{-1}B_1^2,$$

that is to say,

$$P=\begin{pmatrix}0&0&0&\frac{\varsigma}{\gamma^2}\\0&0&\frac{\alpha\varsigma}{\beta\gamma}&0\\0&-\frac{\eta}{\beta^2}&0&0\\-\frac{\alpha\eta}{\beta\gamma}&0&0&0\end{pmatrix}.\tag{3.10}$$

Remembering (3.6), we thus find out that

$$\widehat{V}=\begin{pmatrix}\widehat{\mathfrak{b}}\\\widehat{\mathfrak{d}}\\\widehat{\mathfrak{j}}_0\\\widehat{\mathfrak{j}}_1\end{pmatrix}:=\begin{pmatrix}1&0&0&\frac{\varsigma}{\gamma^2}\rho\\-\frac{\alpha\varsigma\eta}{\beta^2\gamma}\rho&1&\frac{\alpha\varsigma}{\beta\gamma}\rho&\frac{\varsigma}{\gamma}\\-\frac{\eta}{\beta}&-\frac{\eta}{\beta^2}\rho&1&-\frac{\varsigma\eta}{\beta^2\gamma}\rho\\-\frac{\alpha\eta}{\beta\gamma}\rho&0&0&1\end{pmatrix}\begin{pmatrix}\widehat{b}\\\widehat{d}\\\widehat{j}_0\\\widehat{j}_1\end{pmatrix}.\tag{3.11}$$

In terms of coefficients $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ and $\widetilde{\varsigma}$, the above change of variables writes

$$\begin{pmatrix}\widehat{\mathfrak{b}}\\\widehat{\mathfrak{d}}\\\widehat{\mathfrak{j}}_0\\\widehat{\mathfrak{j}}_1\end{pmatrix}:=\begin{pmatrix}1&0&0&\frac{\widetilde{\varsigma}}{\widetilde{\gamma}^2\widetilde{\mathcal{C}}^{3/2}}\rho\\-\frac{\widetilde{\alpha}\widetilde{\varsigma}^2\varepsilon}{\widetilde{\beta}^2\widetilde{\gamma}\widetilde{\mathcal{C}}}\rho&1&\frac{\widetilde{\alpha}\widetilde{\varsigma}}{\widetilde{\beta}\widetilde{\gamma}\widetilde{\mathcal{C}}^{1/2}}\rho&\frac{\widetilde{\varsigma}\varepsilon}{\widetilde{\gamma}\widetilde{\mathcal{C}}^{1/2}}\\-\frac{\widetilde{\varsigma}\varepsilon}{\widetilde{\beta}\widetilde{\mathcal{C}}^{1/2}}&-\frac{\widetilde{\varsigma}}{\widetilde{\beta}^2\widetilde{\mathcal{C}}^{3/2}}\rho&1&-\frac{\widetilde{\varsigma}^2\varepsilon}{\widetilde{\beta}^2\widetilde{\gamma}\widetilde{\mathcal{C}}^2}\rho\\-\frac{\widetilde{\alpha}\widetilde{\varsigma}}{\widetilde{\beta}\widetilde{\gamma}\widetilde{\mathcal{C}}^{1/2}}\rho&0&0&1\end{pmatrix}\begin{pmatrix}\widehat{b}\\\widehat{d}\\\widehat{j}_0\\\widehat{j}_1\end{pmatrix}.$$

Hence, assuming that $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ and $\widetilde{\varsigma}$ are of order 1, and that $\widetilde{\mathcal{C}}\gtrsim 1$, we deduce that the constants appearing when changing $(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)$ to $(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)$, and conversely, may be uniformly bounded for $(\varepsilon,\rho)\in[0,R]^2$ (for any fixed $R>0$).

Next, let us use the explicit form of P to rewrite (3.8). To this end, we have to compute the matrices PB_1, $[P,A_1]$ and A_3. We easily get

$$PB_1=\begin{pmatrix}-\frac{\alpha\varsigma\eta}{\beta\gamma^2}&0&0&0\\0&-\frac{\alpha\varsigma\eta}{\beta^2\gamma}&0&0\\0&0&\frac{\alpha\varsigma\eta}{\beta^2\gamma}&0\\0&0&0&\frac{\alpha\varsigma\eta}{\beta\gamma^2}\end{pmatrix}\quad\text{and}$$

$$[P,A_1]=\begin{pmatrix} 0 & 0 & -\frac{\alpha\varsigma}{\gamma}(\frac{1}{\beta}+\frac{1}{\gamma}) & 0\\ 0 & 0 & 0 & \frac{\alpha\alpha'\varsigma}{\beta\gamma}+\frac{\varsigma}{\gamma^2}(1+\frac{\alpha\varsigma\eta}{\beta\gamma})\\ \frac{\alpha\alpha'\eta}{\beta\gamma}+\frac{\eta}{\beta^2}(1+\frac{\alpha\varsigma\eta}{\beta\gamma}) & 0 & 0 & 0\\ 0 & -\frac{\alpha\eta}{\beta}(\frac{1}{\beta}+\frac{1}{\gamma}) & 0 & 0\end{pmatrix}.$$

Finally, $A_3:=(PA_0-A_1)P^2+\nu A_2P$, and thus

$$A_3=\frac{\alpha\varsigma\eta}{\beta\gamma}\begin{pmatrix} 0 & \frac{1}{\beta^2} & 0 & -\frac{\varsigma}{\gamma^3}\\ \frac{\nu}{\gamma}-\frac{1}{\gamma^2}\big(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\big) & 0 & \frac{\nu}{\eta}-\frac{\alpha\varsigma}{\beta^2\gamma} & 0\\ 0 & 0 & 0 & \frac{\alpha'}{\gamma^2}\\ 0 & 0 & -\frac{\alpha}{\beta^2} & 0\end{pmatrix}. \tag{3.12}$$

Hence

$$|A_3|\leq C(1+\nu) \tag{3.13}$$

with C depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, and $\widetilde{\varsigma}$ (if assuming that $\widetilde{\mathcal{C}}\gtrsim 1$).

Let us focus on the system satisfied by $(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})$ for a while. We have

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\end{pmatrix}+\rho\begin{pmatrix}0 & 1\\ -1-\frac{\alpha\varsigma\eta}{\beta\gamma} & 0\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\end{pmatrix}+\rho^2\begin{pmatrix}-\frac{\alpha\varsigma\eta}{\beta\gamma^2} & 0\\ 0 & \nu-\frac{\alpha\varsigma\eta}{\beta^2\gamma}\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\end{pmatrix}\\ =\rho^2\begin{pmatrix}\frac{\alpha\varsigma}{\gamma}(\frac{1}{\beta}+\frac{1}{\gamma}) & 0\\ 0 & \frac{\varsigma\nu}{\gamma}-\frac{\alpha\alpha'\varsigma}{\beta\gamma}-\frac{\varsigma}{\gamma^2}(1+\frac{\alpha\varsigma\eta}{\beta\gamma})\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix}+(1+\nu)\mathcal{O}(\rho^3).\end{aligned} \tag{3.14}$$

For small enough ρ, optimal estimates may be proved by taking advantage of the results of Appendix D. Indeed, we see from (A.7) and (A.8) that if we set

$$\mathcal{U}_\rho^2:=\Big(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\Big)|\widehat{\mathfrak{b}}|^2+|\widehat{\mathfrak{d}}|^2-\rho\Big(\nu+\frac{\alpha\varsigma\eta}{\beta\gamma}\Big(\frac{1}{\gamma}-\frac{1}{\beta}\Big)\Big)\mathrm{Re}\,(\widehat{\mathfrak{b}}\,\overline{\widehat{\mathfrak{d}}})$$

then, under the following stability condition[3] :

$$\tilde{\nu}:=\nu-\frac{\alpha\varsigma\eta}{\beta\gamma}\Big(\frac{1}{\beta}+\frac{1}{\gamma}\Big)>0, \tag{3.15}$$

we have for small enough ε (see (A.5) and (A.6)):

$$\frac{1}{2}\mathcal{U}_\rho\leq|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\leq 2\mathcal{U}_\rho\quad\text{and}\quad\frac{d}{dt}\mathcal{U}_\rho^2+\frac{\nu}{4}\rho^2\mathcal{U}_\rho^2\leq C\mathcal{U}_\rho|\widehat{F}_\rho|, \tag{3.16}$$

where $\widehat{F}_\rho$ stands for the r.h.s. of (3.14) and C is an absolute constant, whenever

$$\rho\leq\frac{\sqrt{1+\frac{\alpha\varsigma\eta}{\beta\gamma}}}{\nu+\frac{\alpha\varsigma\eta}{\beta\gamma}(\frac{1}{\gamma}-\frac{1}{\beta})}. \tag{3.17}$$

Note that in the case we are interested in, $\gamma\geq\beta$, hence (3.17) is fulfilled if $\rho\nu\leq 1$.

[3]Which is satisfied for $\varepsilon\to 0$ as $\tilde{\nu}\to\nu$.

So finally, we get for small enough ε and some appropriate constant $C=C(\widetilde\alpha,\widetilde\beta,\widetilde\gamma,\widetilde\varsigma)$:

$$|(\widehat{\mathfrak b},\widehat{\mathfrak d})(t)|+\nu\rho^2\int_0^t|(\widehat{\mathfrak b},\widehat{\mathfrak d})|\,d\tau \\ \leq C\biggl(|(\widehat{\mathfrak b},\widehat{\mathfrak d})(0)|+(1+\varepsilon\nu)\rho^2\int_0^t|(\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)|\,d\tau+(1+\nu)\rho^3\int_0^t|(\widehat{\mathfrak b},\widehat{\mathfrak d},\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)|\,d\tau\biggr),$$

which, if in addition $(1+\nu)\rho\ll\nu$, may be simplified into

$$|(\widehat{\mathfrak b},\widehat{\mathfrak d})(t)|+\nu\rho^2\int_0^t|(\widehat{\mathfrak b},\widehat{\mathfrak d})|\,d\tau\leq C\biggl(|(\widehat{\mathfrak b},\widehat{\mathfrak d})(0)|+(1+\varepsilon\nu)\rho^2\int_0^t|(\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)|\,d\tau\biggr)\cdot \tag{3.18}$$

Next we see that the modified radiative modes $\mathfrak j_0$ and $\mathfrak j_1$ satisfy:

$$\frac{d}{dt}\begin{pmatrix}\widehat{\mathfrak j}_0\\ \widehat{\mathfrak j}_1\end{pmatrix}+\rho\begin{pmatrix}0&\alpha'\\-\alpha&0\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak j}_0\\ \widehat{\mathfrak j}_1\end{pmatrix}+\begin{pmatrix}\beta+\frac{\alpha\varsigma\eta}{\beta^2\gamma}\rho^2&0\\0&\gamma+\frac{\alpha\varsigma\eta}{\beta\gamma^2}\rho^2\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak j}_0\\ \widehat{\mathfrak j}_1\end{pmatrix} \\ =\rho^2\begin{pmatrix}-\frac{\alpha\alpha'\eta}{\beta\gamma}-\frac{\eta}{\beta^2}(1+\frac{\alpha\varsigma\eta}{\beta\gamma})&0\\0&\frac{\alpha\eta}{\beta}(\frac1\beta+\frac1\gamma)\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak b}\\ \widehat{\mathfrak d}\end{pmatrix}+(1+\nu)\mathcal O(\rho^3). \tag{3.19}$$

Therefore, there exists some constant $C=C(\widetilde\alpha,\widetilde\beta,\widetilde\gamma,\widetilde\varsigma)$ so that

$$\frac12\frac{d}{dt}\biggl(|\widehat{\mathfrak j}_0|^2+\frac{\alpha'}{\alpha}|\widehat{\mathfrak j}_1|^2\biggr)+\biggl(\beta+\frac{\alpha\varsigma\eta}{\beta^2\gamma}\rho^2\biggr)|\widehat{\mathfrak j}_0|^2+\frac{\alpha'}{\alpha}\biggl(\gamma+\frac{\alpha\varsigma\eta}{\beta\gamma^2}\rho^2\biggr)|\widehat{\mathfrak j}_1|^2 \\ \leq C\bigl((1+\widetilde{\mathcal C}^{1/2}\varepsilon)\rho^2|(\widehat{\mathfrak b},\widehat{\mathfrak d})|+(1+\nu)\rho^3|(\widehat{\mathfrak b},\widehat{\mathfrak d},\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)|\bigr).$$

Then, integrating in time, remembering that $\min(\beta,\gamma)=\min(\widetilde\beta,\widetilde\gamma)\widetilde{\mathcal C}\varepsilon$ and assuming that

$$\widetilde{\mathcal C}^{1/2}\varepsilon\lesssim1,\quad (1+\nu)\rho\ll1\ \text{ and }\ (1+\nu)\rho^3\ll\varepsilon\widetilde{\mathcal C}$$

yields :

$$|(\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)(t)|+\varepsilon\widetilde{\mathcal C}\int_0^t|(\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)|\,d\tau\leq C\biggl(|(\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)(0)|+\rho^2\int_0^t|(\widehat{\mathfrak b},\widehat{\mathfrak d})|\,d\tau\biggr)\cdot \tag{3.20}$$

Combining with (3.18), we can conclude that if ε is small enough then there exists some positive constants ρ_0 and C depending only on $(\widetilde\alpha,\widetilde\beta,\widetilde\gamma,\widetilde\varsigma)$ so that for all

$$0\leq\rho\leq\rho_0\min\biggl(\nu,\nu^{-1},(\varepsilon\nu\widetilde{\mathcal C})^{\frac12},\Bigl(\frac{\varepsilon\widetilde{\mathcal C}}{1+\nu}\Bigr)^{\frac13}\biggr), \tag{3.21}$$

we have

$$|(\widehat{\mathfrak b},\widehat{\mathfrak d})(t)|+\nu|(\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)(t)|+\nu\rho^2\int_0^t|(\widehat{\mathfrak b},\widehat{\mathfrak d})|\,d\tau+\nu\varepsilon\widetilde{\mathcal C}\int_0^t|(\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)|\,d\tau \\ \leq C\bigl(|(\widehat{\mathfrak b},\widehat{\mathfrak d})(0)|+\nu|(\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)(0)|\bigr). \tag{3.22}$$

3.2. **Estimates for large ρ.** The case of large ρ's (high frequency regime) is tractable by means of a more standard approach, as the coupling between (b,d) and (j_0,j_1) is low order. In fact, considering the term in j_1 in the equation for d as a source term, the 'classical' estimate for the linearized barotropic equations gives[4] :

$$|(\widehat b,\nu\rho\widehat b,\widehat d)(t)|+\min(1,\nu\rho)\int_0^t\rho|\widehat b|\,d\tau+\nu\rho^2\int_0^t|\widehat d|\,d\tau\leq C\biggl(|(\widehat b,\nu\rho\widehat b,\widehat d)(0)|+\widetilde{\mathcal C}^{1/2}\varepsilon^2\int_0^t|\widehat j_1|\,d\tau\biggr),$$

[4]That may be proved by considering $|\nu\rho\widehat b|^2+2|(\widehat b,\widehat d)|^2-2\mathrm{Re}\,(\nu\rho\widehat b\,\overline{\widehat d})$, see [1], Chap. 10.

while the equations for $(\widehat{j}_0,\widehat{j}_1)$ give us

$$|(\widehat{j}_0,\widehat{j}_1)(t)|+\widetilde{\mathcal{C}}\varepsilon\int_0^t|(\widehat{j}_0,\widehat{j}_1)|\,d\tau\leq|(\widehat{j}_0,\widehat{j}_1)(0)|+\widetilde{\mathcal{C}}^{1/2}\varepsilon^2\int_0^t|\widehat{b}|\,d\tau.$$

Adding up those two inequalities, we discover that

$$|(\widehat{b},\nu\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)|+\min(1,\nu\rho)\int_0^t\rho|\widehat{b}|\,d\tau+\nu\rho^2\int_0^t|\widehat{d}|\,d\tau$$
$$+\varepsilon\widetilde{\mathcal{C}}\int_0^t|(\widehat{j}_0,\widehat{j}_1)|\,d\tau\leq C|(\widehat{b},\nu\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)|,\tag{3.23}$$

provided (still assuming $1\lesssim\mathcal{C}\lesssim\varepsilon^{-2}$)

$$\rho\gg\frac{\varepsilon\,\widetilde{\mathcal{C}}^{1/4}}{\nu^{1/2}}.\tag{3.24}$$

Note that there is some overlap between the low frequency condition given by (3.21), and (3.24).

For simplicity, we shall assume from now on that (2.4) is fulfilled. This will imply that for any $\nu_0>0$ there exist $\varepsilon_0,\ \rho_0$ and ρ_1 depending only on $\widetilde{\alpha},\ \widetilde{\beta},\ \widetilde{\gamma},\ \widetilde{\varsigma}$ and ν_0 so that (3.22) and (3.23) are fulfilled whenever $\varepsilon\leq\varepsilon_0,\ \nu\leq\nu_0$ and

$$\rho\leq\rho_0\sqrt{\varepsilon\nu\widetilde{\mathcal{C}}}\quad\text{and}\quad\rho\geq\rho_1\nu^{-1/2}\widetilde{\mathcal{C}}^{1/4}\varepsilon,\quad\text{respectively.}\tag{3.25}$$

4. Proof of the global existence

This section is dedicated to the proof of the first part of Theorem 2.1.

Set $\rho_\varepsilon:=\rho_0\sqrt{\varepsilon\nu\widetilde{\mathcal{C}}}$ and introduce the space $E^s_{\varepsilon,\nu,\widetilde{\mathcal{C}}}$ of quadruplets of functions $(b,\vec{u},j_0,\vec{j}_1)$ with

$$\vec{u}\in\mathcal{C}(\mathbb{R}_+;\dot{B}^s_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{s+2}_{2,1}),$$
$$b^{\ell,\nu^{-1}}\in\mathcal{C}(\mathbb{R}_+;\dot{B}^s_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{s+2}_{2,1}),\quad b^{h,\nu^{-1}}\in\mathcal{C}(\mathbb{R}_+;\dot{B}^s_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^s_{2,1}),$$
$$(j_0^{\ell,\rho_\varepsilon},\vec{j}_1^{\,\ell,\rho_\varepsilon})\in\mathcal{C}(\mathbb{R}_+;\dot{B}^s_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{s+2}_{2,1}),$$
$$(j_0^{h,\rho_\varepsilon},\vec{j}_1^{\,h,\rho_\varepsilon})\in\mathcal{C}(\mathbb{R}_+;\dot{B}^s_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^s_{2,1}),$$

such that the following norm is finite:

$$\|(b,\vec{u},j_0,\vec{j}_1)\|_{E^s_{\varepsilon,\nu,\widetilde{\mathcal{C}}}}:=\|\vec{u}\|_{L^\infty(\dot{B}^s_{2,1})}+\nu\|(j_0,\vec{j}_1)\|^{\ell,\rho_\varepsilon}_{L^\infty(\dot{B}^s_{2,1})}+\|(j_0,\vec{j}_1)\|^{h,\rho_\varepsilon}_{L^\infty(\dot{B}^s_{2,1})}$$
$$+\|b\|^{\ell,\nu^{-1}}_{L^\infty(\dot{B}^s_{2,1})}+\nu\|b\|^{h,\nu^{-1}}_{L^\infty(\dot{B}^{s+1}_{2,1})}+\int_{\mathbb{R}_+}\Big(\nu\|\vec{u}\|_{\dot{B}^{s+2}_{2,1}}+\nu\|b\|^{\ell,\nu^{-1}}_{\dot{B}^{s+2}_{2,1}}+\|b\|^{h,\nu^{-1}}_{\dot{B}^{s+1}_{2,1}}\Big)dt$$
$$+\varepsilon\widetilde{\mathcal{C}}\int_{\mathbb{R}_+}\big(\nu\|(\mathrm{j}_0,\vec{\mathrm{j}}_1)\|^{\ell,\rho_\varepsilon}_{\dot{B}^s_{2,1}}+\|(j_0,\vec{j}_1)\|^{h,\rho_\varepsilon}_{\dot{B}^s_{2,1}}\big)\,dt.$$

with

$$\mathrm{j}_0:=j_0-\frac{\widetilde{\varsigma}\varepsilon}{\widetilde{\beta}\widetilde{\mathcal{C}}^{1/2}}b-\frac{\widetilde{\varsigma}}{\widetilde{\beta}^2\widetilde{\mathcal{C}}^{3/2}}\operatorname{div}\vec{u}-\frac{\widetilde{\varsigma}^2\varepsilon}{\widetilde{\beta}^2\widetilde{\gamma}\widetilde{\mathcal{C}}^2}\operatorname{div}\vec{j}_1\quad\text{and}\quad\vec{\mathrm{j}}_1:=\vec{j}_1-\frac{\widetilde{\alpha}\widetilde{\varsigma}}{\widetilde{\beta}\widetilde{\gamma}\widetilde{\mathcal{C}}^{1/2}}\nabla b.\tag{4.1}$$

Through the rescaling (2.8), the first part of Theorem 2.1 is a straightforward consequence of the following statement.

Theorem 4.1. *Let $\nu_0 > 0$ and $n \geq 2$. Assume* (2.4). *There exist three constants ε_0, c and C depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{\varsigma}$, ν_0 and λ/μ such that if $0 < \varepsilon \leq \varepsilon_0$, $0 < \nu \leq \nu_0$ and the data satisfy*

$$
\begin{aligned}
(4.2)\quad I_0 := \|\vec{u_0}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} &+ \nu\|(j_{0,0}, \vec{j}_{1,0})\|^{\ell,\rho_\varepsilon}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&+ \|(j_{0,0}, \vec{j}_{1,0})\|^{h,\rho_\varepsilon}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|b_0\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu\|b_0\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} \leq c\nu
\end{aligned}
$$

then System (2.9) *has a unique global solution $(b, \vec{u}, j_0, \vec{j}_1)$ in $E^{\frac{n}{2}-1}_{\varepsilon,\nu,\widetilde{\mathcal{C}}}$ satisfying*

$$
\|(b, \vec{u}, j_0, \vec{j}_1)\|_{E^{\frac{n}{2}-1}_{\varepsilon,\nu,\widetilde{\mathcal{C}}}} \leq C I_0.
$$

Proof of Theorem 2.1: Granted with the above statement, making the change of unknowns (2.8) gives the first part of Theorem 2.1. Indeed, we notice that, up to some irrelevant constant depending only on σ_a, σ_s, $\bar{\varrho}$, $P'(\bar{\varrho})$ and n the term I_0^ε is equal to the quantity I_0 of Theorem 4.1. Besides, computing $\|(b, \vec{u}, j_0, \vec{j}_1)\|_{E^{\frac{n}{2}-1}_{\varepsilon,\nu,\widetilde{\mathcal{C}}}}$ in terms of $(b^\varepsilon, \vec{u}^\varepsilon, j_0^\varepsilon, \vec{j}_1^\varepsilon)$ gives (2.7). □

The rest of this section is devoted to the proof of Theorem 4.1. We focus on the proof of global a priori estimates for smooth solutions to (2.9), and refer to [7] for existence and uniqueness.

At a first trial, one may tempt to apply the analysis of the previous section to System (3.1) with

$$
f = -\vec{u}\cdot\nabla b + k_1(b)\mathrm{div}\,\vec{u} \quad \text{and} \quad \vec{g} = -\vec{u}\cdot\nabla\vec{u} + \nu k_2(b)\widetilde{\mathcal{A}}\vec{u} + k_3(b)\nabla b + \widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2 k_4(b)\vec{j}_1.
$$

Indeed, localizing System (3.1) in the Fourier space by means of the Littlewood-Paley operator $\dot{\Delta}_k$, and combining the inequalities that we proved in Section 3 with Fourier-Plancherel theorem, it is easy to deduce estimates in any Besov space related to L^2. Nonzero terms f and $\vec{g}$ may be included in our analysis, by taking advantage of Duhamel formula. However, one cannot treat the convection term $\vec{u}\cdot\nabla b$ as a source, as it would cause a loss of one derivative (exactly as for the standard compressible Navier-Stokes equations). At the L^2 level, this may be avoided by an energy method, after integrating by parts in that term. In our case, the idea is more or less the same, except that the energy argument has to be performed on the localized convection term, namely $\dot{\Delta}_k(\vec{u}\cdot\nabla b)$. A nice (and nowadays classical) way of performing this computation is to *paralinearize* the system: we consider

$$
(4.3)\qquad \begin{cases}
\partial_t b + T_{\vec{v}}\cdot\nabla b + \mathrm{div}\,\vec{u} = f,\\
\partial_t\vec{u} + T_{\vec{v}}\cdot\nabla\vec{u} + \nabla b - \nu\widetilde{\mathcal{A}}\vec{u} - \widetilde{\mathcal{C}}^{1/2}\widetilde{\varsigma}\varepsilon^2\vec{j}_1 = \vec{g},\\
\partial_t j_0 + \widetilde{\beta}\widetilde{\mathcal{C}}\varepsilon j_0 + \widetilde{\alpha}\widetilde{\mathcal{C}}\mathrm{div}\,\vec{j}_1 - \widetilde{\mathcal{C}}^{1/2}\widetilde{\varsigma}\varepsilon^2 b = 0,\\
\partial_t\vec{j}_1 + \widetilde{\gamma}\widetilde{\mathcal{C}}\varepsilon\vec{j}_1 + \widetilde{\alpha}\widetilde{\mathcal{C}}\nabla j_0 = \vec{0},
\end{cases}
$$

where the time dependent transport vector field $\vec{v}$ and the source terms $f, \vec{g}$ are given.

4.1. **Estimates for the paralinearized system.** Before stating the estimates, let us introduce one more notation: we shall need the following quantity involving medium frequencies:

$$\|z\|^{m,\eta,\iota}_{\dot B^s_{2,1}} := \sum_{\eta/2\leq 2^k\leq 2\iota} 2^{ks}\|\dot\Delta_k z\|_{L^2}.$$

Proposition 4.1. *Let $\nu_0>0$. Assume that (2.4) is fulfilled. There exist positive constants $\varepsilon_0=\varepsilon_0(\widetilde\alpha,\widetilde\beta,\widetilde\gamma,\widetilde\varsigma,\nu_0)$ and $C=C(\widetilde\alpha,\widetilde\beta,\widetilde\gamma,\widetilde\varsigma,\lambda/\mu,\nu_0)$ such that if $0<\nu\leq\nu_0$ and $0<\varepsilon\leq\varepsilon_0$ then the solutions to (4.3) satisfy the following a priori estimates (for all $s\in\mathbb{R}$) :*

- *Low frequencies:*

$$\|(b,\vec u)(t)\|^{\ell,\rho_\varepsilon}_{\dot B^s_{2,1}}+\nu\|(j_0,\vec j_1)(t)\|^{\ell,\rho_\varepsilon}_{\dot B^s_{2,1}}+\nu\int_0^t\Big(\|(b,\vec u)\|^{\ell,\rho_\varepsilon}_{\dot B^{s+2}_{2,1}}+\varepsilon\widetilde{\mathcal C}\|(\mathfrak{j}_0,\vec{\mathfrak{j}}_1)\|^{\ell,\rho_\varepsilon}_{\dot B^s_{2,1}}\Big)d\tau$$
$$\leq C\bigg(\|(b,\vec u)(0)\|^{\ell,\rho_\varepsilon}_{\dot B^s_{2,1}}+\nu\|(j_0,\vec j_1)(0)\|^{\ell,\rho_\varepsilon}_{\dot B^s_{2,1}}+\int_0^t\|(f-T_{\vec v}\cdot\nabla b,\vec g-T_{\vec v}\cdot\nabla\vec u)\|^{\ell,\rho_\varepsilon}_{\dot B^s_{2,1}}\,d\tau\bigg)\cdot$$

- *Middle frequencies:*

$$\|(b,\vec u,j_0,\vec j_1)(t)\|^{m,\rho_\varepsilon,1/\nu}_{\dot B^s_{2,1}}+\int_0^t\Big(\nu\|(b,\vec u)\|^{m,\rho_\varepsilon,1/\nu}_{\dot B^{s+2}_{2,1}}+\varepsilon\widetilde{\mathcal C}\|(j_0,\vec j_1)\|^{m,\rho_\varepsilon,1/\nu}_{\dot B^s_{2,1}}\Big)\,d\tau$$
$$\leq C\bigg(\|(b,\vec u,j_0,\vec j_1)(0)\|^{m,\rho_\varepsilon,1/\nu}_{\dot B^s_{2,1}}+\int_0^t\|(f,\vec g)\|^{m,\rho_\varepsilon,1/\nu}_{\dot B^s_{2,1}}\,d\tau$$
$$+\int_0^t\|\nabla\vec v\|_{L^\infty}\|(b,\vec u,j_0,\vec j_1)\|_{\dot B^s_{2,1}}\,d\tau\bigg)\cdot$$

- *High frequencies:*

$$\|(\nu\nabla b,\vec u,j_0,\vec j_1)(t)\|^{h,1/\nu}_{\dot B^s_{2,1}}+\int_0^t\Big(\nu\|\vec u\|^{h,1/\nu}_{\dot B^{s+2}_{2,1}}+\|\nabla b\|^{h,1/\nu}_{\dot B^s_{2,1}}+\widetilde{\mathcal C}\varepsilon\|(j_0,\vec j_1)\|^{h,1/\nu}_{\dot B^s_{2,1}}\Big)\,d\tau$$
$$\leq C\bigg(\|(\nu\nabla b,\vec u,j_0,\vec j_1)(t)\|^{h,1/\nu}_{\dot B^s_{2,1}}+\int_0^t\|(\nu\nabla f,\vec g)\|^{h,1/\nu}_{\dot B^s_{2,1}}\,d\tau$$
$$+\int_0^t\|\nabla\vec v\|_{L^\infty}\|(\nu\nabla b,\vec u,j_0,\vec j_1)\|_{\dot B^s_{2,1}}\,d\tau\bigg)\cdot$$

Proof: The main idea is to localize System (4.3) according to Littlewood-Paley decomposition, and to use Fourier Plancherel theorem to evaluate the L^2 norm of each localized part of the solution. Concretely, applying $\dot\Delta_k$ to (4.3) and denoting $b_k:=\dot\Delta_k b$, $\vec u_k:=\dot\Delta_k\vec u$, etc, gives

$$\left\{\begin{aligned}&\partial_t b_k+\dot\Delta_k(T_{\vec v}\cdot\nabla b)+\mathrm{div}\,\vec u_k=f_k,\\&\partial_t\vec u_k+\dot\Delta_k(T_{\vec v}\cdot\nabla\vec u)+\nabla b_k-\nu\widetilde{\mathcal A}\vec u_k-\widetilde\varsigma\widetilde{\mathcal C}^{1/2}\varepsilon^2\vec j_{1,k}=\vec g_k,\\&\partial_t j_{0,k}+\widetilde\beta\widetilde{\mathcal C}\varepsilon j_{0,k}+\widetilde\alpha\widetilde{\mathcal C}\mathrm{div}\,\vec j_{1,k}-\widetilde\varsigma\widetilde{\mathcal C}^{1/2}\varepsilon^2 b_k=0,\\&\partial_t\vec j_{1,k}+\widetilde\gamma\widetilde{\mathcal C}\varepsilon\vec j_{1,k}+\widetilde{\mathcal C}\widetilde\alpha\nabla j_{0,k}=\vec 0.\end{aligned}\right.\tag{4.4}$$

To handle low frequencies (i.e. $2^k\leq\rho_\varepsilon$), we put the paraconvection terms in the right-hand side of (4.4) and follow the computations leading to (3.22). We thus have to consider $\mathfrak{b}_k:=\dot\Delta_k b$, $\vec{\mathfrak{u}}_k:=\dot\Delta_k\mathfrak{u}$, $\mathfrak{j}_{0,k}:=\dot\Delta_k\mathfrak{j}_0$, $\vec{\mathfrak{j}}_{1,k}:=\dot\Delta_k\vec{\mathfrak{j}}_1$ with $\mathfrak{j}_0$, $\vec{\mathfrak{j}}_1$ defined in (4.1),

$$\mathfrak{b}:=b+\frac{\widetilde\varsigma}{\widetilde\gamma^2\widetilde{\mathcal C}^{3/2}}\mathrm{div}\,\vec j_1\quad\text{and}\quad\vec{\mathfrak{u}}:=\vec u-\frac{\widetilde\alpha\widetilde\varsigma^2\varepsilon}{\widetilde\beta^2\widetilde\gamma\widetilde{\mathcal C}}\nabla b+\frac{\widetilde\alpha\widetilde\varsigma}{\widetilde\beta\widetilde\gamma\widetilde{\mathcal C}^{1/2}}\nabla j_0+\frac{\widetilde\varsigma\varepsilon}{\widetilde\gamma\widetilde{\mathcal C}^{1/2}}\vec j_1.\tag{4.5}$$

Using eventually Fourier-Plancherel theorem, we get for all positive t and $2^k\leq\rho_\varepsilon$

$$\|(\mathfrak{b}_k,\vec{\mathfrak{u}}_k)(t)\|_{L^2}+\nu\|(\mathfrak{j}_{0,k},\vec{\mathfrak{j}}_{1,k})(t)\|_{L^2}+2^{2k}\nu\int_0^t\|(\mathfrak{b}_k,\vec{\mathfrak{u}}_k)\|_{L^2}\,d\tau+\nu\varepsilon\widetilde{\mathcal{C}}\int_0^t\|(\mathfrak{j}_{0,k},\vec{\mathfrak{j}}_{1,k})\|_{L^2}\,d\tau$$
$$\leq C\biggl(\|(\mathfrak{b}_k,\vec{\mathfrak{u}}_k)(0)\|_{L^2}+\nu\|(\mathfrak{j}_{0,k},\vec{\mathfrak{j}}_{1,k})(0)\|_{L^2}$$
$$+\int_0^t\bigl(\|f_k-\dot\Delta_k(T_{\vec v}\cdot\nabla b)\|_{L^2}+\|\vec g_k-\dot\Delta_k(T_{\vec v}\cdot\nabla\vec u)\|_{L^2}\bigr)\,d\tau\biggr)\cdot$$

Note that, as $2^k\leq\rho_\varepsilon$, the first two terms of the l.h.s. and of the r.h.s. may be changed to the similar ones with $(b_k,\vec u_k,j_{0,k},\vec j_{1,k})$ (up to a harmless change of the constant C of course). Hence multiplying by 2^{ks} and summing up over all $k\in\mathbb{Z}$ with $2^k\leq\rho_\varepsilon$ yields the desired inequality.

To handle the regime corresponding to $2^k\geq\rho_1\nu^{-1/2}\widetilde{\mathcal{C}}^{1/4}\varepsilon$, we introduce the Lyapunov functional

$$L_k^2:=2\|(b_k,\vec u_k)\|_{L^2}^2+\|\nu\nabla b_k\|_{L^2}^2+2\nu(\nabla b_k|\vec u_k),$$

which is obviously equivalent to $\|(b_k,\nu\nabla b_k,\vec u_k)\|_{L^2}^2$. We observe that

$$\frac12\frac{d}{dt}L_k^2+\nu\|(\nabla b_k,\mathcal{Q}\vec u_k)\|_{L^2}^2+2\mu\|\nabla\mathcal{P}\vec u_k\|_{L^2}^2=2\varepsilon^2\widetilde{\mathcal{C}}^{1/2}\widetilde{\varsigma}(\vec u_k|\vec j_{1,k})$$
$$+\varepsilon^2\widetilde{\mathcal{C}}^{1/2}\widetilde{\varsigma}(\nu\nabla b_k|\vec j_{1,k})+\varepsilon^2\widetilde{\mathcal{C}}^{1/2}\widetilde{\varsigma}(b_k|j_{0,k})+2(f_k|b_k)+2(\vec g_k|\vec u_k)+(\nu\nabla f_k|\nu\nabla b_k)+(\nu\nabla f_k|\vec u_k)$$
$$+(\vec g_k|\nu\nabla b_k)+2\bigl(\dot\Delta_k(T_{\vec v}\cdot\nabla b)|b_k\bigr)+2\bigl(\dot\Delta_k(T_{\vec v}\cdot\nabla\vec u)|\vec u_k\bigr)+\bigl(\nu\nabla\dot\Delta_k(T_{\vec v}\cdot\nabla b)|\nu\nabla b_k\bigr)$$
$$+\bigl(\nu\nabla\dot\Delta_k(T_{\vec v}\cdot\nabla b)|\vec u_k\bigr)+\bigl(\dot\Delta_k(T_{\vec v}\cdot\nabla\vec u)|\nu\nabla b_k\bigr).$$

Convection terms may be handled according to Lemma 4.1 in [7]: for example, we have for some universal integer N:

$$\bigl|\bigl(\nabla\dot\Delta_k(T_{\vec v}\cdot\nabla b)|\nabla b_k\bigr)\bigr|\leq C\|\nabla\vec v\|_{L^\infty}\|\nabla b_k\|_{L^2}\sum_{|k'-k|\leq N}\|\nabla b_{k'}\|_{L^2}.$$

We thus eventually get:

$$\frac12\frac{d}{dt}L_k^2+\min(\nu2^{2k},1)L_k^2\leq CL_k\biggl(\|(f_k,\vec g_k,\nu\nabla f_k)\|_{L^2}+\widetilde{\mathcal{C}}^{1/2}\widetilde{\varsigma}\varepsilon^2\|\vec j_{1,k}\|_{L^2}+\|\nabla\vec v\|_{L^\infty}\sum_{|k'-k|\leq N}L_{k'}\biggr)$$

whence, integrating in time,

$$L_k(t)+\min(\nu2^{2k},1)\int_0^tL_k(\tau)\,d\tau\leq L_k(0)+C\int_0^t\|(f_k,\vec g_k,\nu\nabla f_k)\|_{L^2}\,d\tau$$
$$+C\widetilde{\mathcal{C}}^{1/2}\widetilde{\varsigma}\varepsilon^2\int_0^t\|\vec j_{1,k}\|_{L^2}\,d\tau+C\sum_{|k'-k|\leq N}\int_0^t\|\nabla\vec v\|_{L^\infty}L_{k'}\,d\tau.\tag{4.6}$$

From the last two equations of (4.4), we easily get (remembering that $\widetilde\gamma\geq\widetilde\beta$):

$$\|(j_{0,k},\vec j_{1,k})(t)\|_{L^2}+\widetilde{\mathcal{C}}\varepsilon\widetilde\beta\int_0^t\|(j_{0,k},\vec j_{1,k})\|_{L^2}\,d\tau\leq\|(j_{0,k},\vec j_{1,k})(t)\|_{L^2}+\widetilde{\mathcal{C}}^{1/2}\widetilde{\varsigma}\varepsilon^2\int_0^t\|b_k\|_{L^2}\,d\tau.$$

Combining with (4.6) and taking ρ_1 large enough, we conclude that for small enough ε and any $\nu \le \nu_0$,

$$\begin{aligned}(4.7)\quad &\|(b_k, \nu\nabla b_k, \vec{u}_k, j_{0,k}, \vec{j}_{1,k})(t)\|_{L^2} + \min(\nu 2^{2k}, 1)\int_0^t \|(b_k, \nu\nabla b_k, \vec{u}_k)\|_{L^2}\,d\tau \\ &+ \widetilde{\mathcal{C}}\varepsilon\int_0^t \|(j_{0,k}, \vec{j}_{1,k})\|_{L^2}\,d\tau \le C\bigg(\|(b_k, \nu\nabla b_k, \vec{u}_k, j_{0,k}, \vec{j}_{1,k})(0)\|_{L^2} \\ &+ \int_0^t \|(f_k, \vec{g}_k, \nu\nabla f_k)\|_{L^2}\,d\tau + C\sum_{|k'-k|\le N}\int_0^t \|\nabla\vec{v}\|_{L^\infty}\|(b_{k'}, \nu\nabla b_{k'}, \vec{u}_{k'}, j_{0,k'}, \vec{j}_{1,k'})\|_{L^2}\,d\tau\bigg)\cdot\end{aligned}$$

In order to exhibit the parabolic gain of regularity for $\vec{u}$, we use the fact that

$$\partial_t\vec{u}_k - \nu\widetilde{\mathcal{A}}\vec{u}_k = \vec{g}_k - \nabla b_k + \widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2\vec{j}_{1,k} - \dot{\Delta}_k(T_{\vec{v}}\cdot\nabla\vec{u}),$$

which implies by energy method, Bernstein inequality and Lemma 4.1 in [7],

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\vec{u}_k\|_{L^2}^2 + \nu 2^{2k}\|\vec{u}_k\|_{L^2}^2 \le C\|\vec{u}_k\|_{L^2}\bigg(&\|\vec{g}_k\|_{L^2} + \varepsilon^2\widetilde{\mathcal{C}}^{1/2}\|\vec{j}_{1,k}\|_{L^2} \\ &+\|\nabla b_k\|_{L^2} + \sum_{|k'-k|\le N}\|\nabla\vec{v}\|_{L^\infty}\|\vec{u}_{k'}\|_{L^2}\bigg)\cdot\end{aligned}$$

Integrating with respect to time, and keeping (4.7) in mind, one may bound the second and third term of the r.h.s. in terms of the data. We thus conclude that

$$\begin{aligned}&\|(b_k, \nu\nabla b_k, \vec{u}_k, j_{0,k}, \vec{j}_{1,k})(t)\|_{L^2} + \min(1, \nu 2^k)\int_0^t \|\nabla b_k\|_{L^2}\,d\tau + \nu 2^{2k}\int_0^t \|\vec{u}_k\|_{L^2}\,d\tau \\ &+\widetilde{\mathcal{C}}\varepsilon\int_0^t \|(j_{0,k}, \vec{j}_{1,k})\|_{L^2}\,d\tau \le C\bigg(\|(b_k, \nu\nabla b_k, \vec{u}_k, j_{0,k}, \vec{j}_{1,k})(0)\|_{L^2} \\ &+ \int_0^t \|(f_k, \vec{g}_k, \nu\nabla f_k)\|_{L^2}\,d\tau + C\sum_{|k'-k|\le N}\int_0^t \|\nabla\vec{v}\|_{L^\infty}\|(b_{k'}, \nu\nabla b_{k'}, \vec{u}_{k'}, j_{0,k'}, \vec{j}_{1,k'})\|_{L^2}\,d\tau\bigg)\cdot\end{aligned}$$

Multiplying by 2^{ks} and summing up over all the integers $k \in \mathbb{Z}$ such that $2^k \ge \rho_1\nu^{-1/2}\widetilde{\mathcal{C}}^{1/4}\varepsilon$ yields the wanted inequalities in the middle and high frequencies range. $\square$

4.2. **Uniform estimates.** It is now easy to prove global estimates for smooth solutions to (2.9) : it suffices to apply Proposition 4.1 with $s = n/2 - 1$,

$$f = -T'_{\nabla b}\cdot\vec{u} + k_1(b)\mathrm{div}\,\vec{u} \quad\text{and}\quad \vec{g} = -T'_{\nabla\vec{u}}\cdot\vec{u} + \nu k_2(b)\widetilde{\mathcal{A}}\vec{u} + k_3(b)\nabla b + \widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2 k_4(b)\vec{j}_1.$$

As regards f, standard product, paraproduct and composition estimates in Besov spaces lead to

$$\begin{aligned}(4.8)\quad \|f\|_{L^1(\dot{B}^{\frac{n}{2}-1}_{2,1}\cap\dot{B}^{\frac{n}{2}}_{2,1})} \lesssim\; &\|\nabla b\|_{L^\infty(\dot{B}^{\frac{n}{2}-2}_{2,1}\cap\dot{B}^{\frac{n}{2}-1}_{2,1})}\|\vec{u}\|_{L^1(\dot{B}^{\frac{n}{2}+1}_{2,1})} \\ &+\|b\|_{L^\infty(\dot{B}^{\frac{n}{2}-1}_{2,1}\cap\dot{B}^{\frac{n}{2}}_{2,1})}\|\mathrm{div}\,\vec{u}\|_{L^1(\dot{B}^{\frac{n}{2}}_{2,1})}.\end{aligned}$$

Therefore, if $\nu=1$ then Inequality (4.8) provides a control of the left-hand side by $\|(b,\vec u,j_0,\vec j_1)\|^2_{E^{\frac n2-1}_{\varepsilon,1,\widetilde{\mathcal C}}}$. The general case follows from the case $\nu=1$ after change of variables, and we eventually get

$$
\begin{aligned}
\|f\|^{\ell,\nu^{-1}}_{L^1(\dot B^{\frac n2-1}_{2,1})}+\nu\|f\|^{h,\nu^{-1}}_{L^1(\dot B^{\frac n2}_{2,1})}&\lesssim\big(\|b\|^{\ell,\nu^{-1}}_{L^\infty(\dot B^{\frac n2-1}_{2,1})}+\nu\|b\|^{h,\nu^{-1}}_{L^\infty(\dot B^{\frac n2}_{2,1})}\big)\|\mathrm{div}\,\vec u\|_{L^1(\dot B^{\frac n2}_{2,1})}\\
&\lesssim\nu^{-1}\|(b,\vec u,j_0,\vec j_1)\|^2_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal C}}}.
\end{aligned}\tag{4.9}
$$

Next, we need a quadratic estimate for $\vec g$ in $L^1(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})$. The first three terms may be handled by means of usual product, paraproduct and composition estimates: we get

$$
\begin{aligned}
&\|T'_{\nabla\vec u}\vec u\|_{L^1(\dot B^{\frac n2-1}_{2,1})}\lesssim\|\nabla\vec u\|_{L^1(\dot B^{\frac n2}_{2,1})}\|\vec u\|_{L^\infty(\dot B^{\frac n2-1}_{2,1})},\\
&\nu\|k_2(b)\widetilde{\mathcal A}\vec u\|_{L^1(\dot B^{\frac n2-1}_{2,1})}\lesssim\nu\|b\|_{L^\infty(\dot B^{\frac n2}_{2,1})}\|\nabla^2\vec u\|_{L^1(\dot B^{\frac n2-1}_{2,1})},\\
&\|k_3(b)\nabla b\|_{L^1(\dot B^{\frac n2-1}_{2,1})}\lesssim\|b\|^2_{L^2(\dot B^{\frac n2}_{2,1})}.
\end{aligned}
$$

Because

$$
\|b\|^{\ell,\nu^{-1}}_{B^{\frac n2-1}_{2,1}}+\nu\|b\|^{h,\nu^{-1}}_{B^{\frac n2}_{2,1}}\approx\|b\|_{\dot B^{\frac n2-1}_{2,1}}+\nu\|b\|_{\dot B^{\frac n2}_{2,1}}
$$

and, by interpolation,

$$
\nu^{1/2}\|b\|_{L^2(\dot B^{\frac n2}_{2,1})}\lesssim\|(b,\vec u,j_0,\vec j_1)\|^2_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal C}}},\tag{4.10}
$$

we deduce that the above three terms may be bounded by $\nu^{-1}\|(b,\vec u,j_0,\vec j_1)\|^2_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal C}}}$.

For the last term, we just write that

$$
k_4(b)\vec j_1=k_4(b)\vec{\mathrm j}_1+\frac{\widetilde\alpha\widetilde\varsigma}{\widetilde\beta\widetilde\gamma\widetilde{\mathcal C}^{1/2}}k_4(b)\nabla b.
$$

Hence

$$
\|k_4(b)\vec j_1\|_{L^1(\dot B^{\frac n2-1}_{2,1})}\lesssim\|b\|_{L^\infty(\dot B^{\frac n2}_{2,1})}\|\vec{\mathrm j}_1\|_{L^1(\dot B^{\frac n2-1}_{2,1})}+\|b\|^2_{L^2(\dot B^{\frac n2}_{2,1})}.\tag{4.11}
$$

Finally, we need to bound the low frequencies of the paraconvection terms $T_{\vec u}\cdot\nabla b$ and $T_{\vec u}\cdot\nabla\vec u$ in $L^1(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})$. We observe (by product laws and interpolation) that

$$
\begin{aligned}
&\|T_{\vec u}\cdot\nabla b\|_{L^1(\dot B^{\frac n2-1}_{2,1})}\lesssim\|\vec u\|_{L^2(\dot B^{\frac n2}_{2,1})}\|\nabla b\|_{L^2(\dot B^{\frac n2-1}_{2,1})}\lesssim\nu^{-1}\|(b,\vec u,j_0,\vec j_1)\|^2_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal C}}},\\
&\|T_{\vec u}\cdot\nabla\vec u\|_{L^1(\dot B^{\frac n2-1}_{2,1})}\lesssim\|\vec u\|_{L^2(\dot B^{\frac n2}_{2,1})}\|\nabla\vec u\|_{L^2(\dot B^{\frac n2-1}_{2,1})}\lesssim\nu^{-1}\|(b,\vec u,j_0,\vec j_1)\|^2_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal C}}}.
\end{aligned}
$$

Putting together the above inequalities and Proposition 4.1 with $s=n/2-1$ and using the embedding $\dot B^{\frac n2}_{2,1}\hookrightarrow L^\infty$, we end up with

$$
\|(b,\vec u,j_0,\vec j_1)\|_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal C}}}\leq C\big(I_0+\nu^{-1}\|(b,\vec u,j_0,\vec j_1)\|^2_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal C}}}\big)\cdot
$$

Assuming that (4.2) is fulfilled with a sufficiently small constant c, it is clear that the above inequality implies that

$$
\|(b,\vec u,j_0,\vec j_1)\|_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal C}}}\leq 2CI_0,
$$

which is exactly what we wanted. □

Remark 4.1. *The above theorem and the following corollary extend to the case where* λ, μ, σ_a *and* σ_s *depend smoothly on* ϱ. *Compared to the constant case, the main difference is that we have to bound in* $L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$ *extra nonlinear terms like* $\nabla(K(b))\otimes\nabla u$, $K_1(b)j_0$, $K_2(b)\vec{j}_1$ *and* $K_3(b)b$ *for some explicit smooth functions* K, K_1, K_2 *and* K_3 *vanishing at* 0. *All those terms may be handled by taking advantage of the damped modes* $\mathfrak{j}_0$ *and* $\vec{\mathfrak{j}}_1$. *However, as we do not know how to treat nonlinear terms of the type* $K(b)b$ *if they occur in the equation for* j_0, *one has to change the definition of* b *into, say,*

$$\widetilde{b} := \frac{\sigma_a(\varrho)\big(B(\varrho)-B(\bar\varrho)\big)}{\sigma_a(\bar\varrho)\,\bar\varrho\,B'(\bar\varrho)}\cdot$$

The details are left to the reader.

5. The proof of convergence

This section is devoted to the rigorous justification of the low Mach number asymptotics pointed out in the introduction. We here propose two approaches. The first one is based on the uniform estimates of Theorem 2.1 and compactness arguments, and is thus valid indistinctly in $\mathbb{T}^n$ or $\mathbb{R}^n$. In contrast, the second approach combines the estimates of Theorem 2.1 with Strichartz type inequalities, and thus works only in the $\mathbb{R}^n$ case with $n\geq 2$. At the same time, the result is more accurate: we get strong convergence for explicit norms and with explicit decay rate.

5.1. Weak convergence results.

Here we establish a general weak convergence result which holds true both in the periodic and the whole space cases.

Theorem 5.1. *Consider a family of data* $(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec{j}_{1,0}^\varepsilon)$ *satisfying the assumptions of Theorem 2.1 with* $\varepsilon\to0$. *Let* $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ *be the corresponding family of global solutions to* (2.1). *Then, for suitable norms,* $\varrho^\varepsilon := B^{-1}(B(\bar\varrho)+\bar\varrho B'(\bar\varrho)\widetilde\varepsilon b^\varepsilon)$ *converges strongly to* $\bar\varrho$, $(j_0^\varepsilon,\vec j_1^\varepsilon)\to(0,\vec 0)$ *with the rate of convergence* $\mathcal{O}(\varepsilon)$, *and* $(b^\varepsilon,\operatorname{div}\vec u^\varepsilon)\rightharpoonup(0,\vec 0)$ *in* $\mathcal{S}'$.

If we suppose in addition that

$$\mathcal{P}\vec u_0^\varepsilon \rightharpoonup \vec v_0 \quad in \quad \mathcal{S}' \tag{5.1}$$

then $\vec u^\varepsilon$ *converges to* $\vec v$ *in* $\mathcal{S}'$ *when* $\varepsilon\to0$, *where* $\vec v\in\mathcal{C}(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac n2+1}_{2,1})$ *stands for the unique global solution of*

$$\begin{cases}\partial_t\vec v+\vec v\cdot\nabla\vec v+\nabla\Pi-\bar\mu\Delta\vec v=\vec 0,\\ \operatorname{div}\vec v=0.\\ \vec v|_{t=0}=\vec v_0,\end{cases}\qquad with\quad \bar\mu:=\mu/\bar\varrho. \tag{5.2}$$

Proof: Let us first observe that

$$\|b_0^\varepsilon\|^{h,\frac{1}{\varepsilon\nu}}_{\dot B^{\frac n2-1}_{2,1}} \lesssim \varepsilon\nu\|b_0^\varepsilon\|^{h,\frac{1}{\varepsilon\nu}}_{\dot B^{\frac n2}_{2,1}}. \tag{5.3}$$

Hence the data $(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)$ are uniformly bounded in $\dot B^{\frac n2-1}_{2,1}$ and one thus have, up to some omitted extraction,

$$(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)\rightharpoonup(b_0,\vec u_0,j_{0,0},\vec j_{1,0})\quad\text{in}\quad \dot B^{\frac n2-1}_{2,1}\quad\text{weak }*\,.$$

Of course, owing to the convergence assumption (5.1), we have $\mathcal{P}\vec u_0=\vec v_0$.

Likewise, the corresponding sequence of solutions $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ is bounded in the space $\mathcal{C}_b(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$ hence we have, up to another omitted extraction,

$$(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)\rightharpoonup(b,\vec u,j_0,\vec j_1)\quad\text{in}\quad L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})\quad\text{weak }*.$$

Let us first focus on $(j_0,\vec j_1)$. The above bounds and convergence result imply that one may pass to the limit in the last two equations of (2.1) in the distributional meaning: we get

$$\frac1n\mathrm{div}\,\vec j_1+\sigma_a(j_0-\bar\varrho B'(\bar\varrho)b)=0\quad\text{and}\quad\nabla j_0+(\sigma_a+\sigma_s)\vec j_1=\vec 0.\tag{5.4}$$

We claim that $(j_0,\vec j_1)\equiv(0,\vec 0)$. Indeed, using the decompositions

$$\begin{aligned}j_0^\varepsilon&=(j_0^\varepsilon)^{h,\widetilde\rho_\varepsilon}+(\mathrm{j}_0^\varepsilon)^{\ell,\widetilde\rho_\varepsilon}+c_1\varepsilon(b^\varepsilon)^{\ell,\widetilde\rho_\varepsilon}+c_2\widetilde{\mathcal C}^{-1}\varepsilon(\mathrm{div}\,\vec u^\varepsilon)^{\ell,\widetilde\rho_\varepsilon}+c_3\widetilde{\mathcal C}^{-2}\varepsilon^2(\mathrm{div}\,\vec j_1^\varepsilon)^{\ell,\widetilde\rho_\varepsilon},\\ \vec j_1^\varepsilon&=(j_1^\varepsilon)^{\ell,\widetilde\rho_\varepsilon}+c_4\varepsilon(\nabla b^\varepsilon)^{\ell,\widetilde\rho_\varepsilon},\end{aligned}$$

the uniform bounds of Theorem 2.1, and (4.10), we deduce that there exists some positive real number M_0 depending only on the coefficients of the system (other than ε and $\widetilde{\mathcal C}$) such that

$$\|j_0^\varepsilon\|_{L^1(\dot B^{\frac n2-1}_{2,1}+\dot B^{\frac n2}_{2,1})+L^2(\dot B^{\frac n2}_{2,1})+L^\infty(\dot B^{\frac n2-2}_{2,1})}+\|\vec j_1^\varepsilon\|_{(L^1+L^2)(\dot B^{\frac n2-1}_{2,1})}\le M_0\varepsilon.\tag{5.5}$$

Hence one may conclude that $j_0\equiv0$ and $\vec j_1\equiv\vec 0$.

The strong convergence of the density to $\bar\varrho$ is obvious: we have

$$\varrho^\varepsilon=B^{-1}\big(B(\bar\varrho)+\bar\varrho B'(\bar\varrho)\widetilde\varepsilon b^\varepsilon\big),$$

hence, given that b^ε is bounded in $L^2(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$, we have $\varrho^\varepsilon\to\bar\varrho$ in $L^2(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$, with rate ε. We even have a stronger result: because $j_0=\mathrm{div}\,\vec j_1=0$, the first equality in (5.4) implies that $b\equiv0$.

To see that $\mathrm{div}\,\vec u=0$, we use the mass equation:

$$\mathrm{div}\,\vec u^\varepsilon=k_1(\widetilde\varepsilon b^\varepsilon)\mathrm{div}\,\vec u^\varepsilon-\widetilde\varepsilon\vec u^\varepsilon\cdot\nabla b^\varepsilon-\widetilde\varepsilon\partial_t b^\varepsilon.$$

Given that b^ε and $\vec u^\varepsilon$ are bounded in $L^2(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$, the first two terms in the right-hand side are $\mathcal{O}(\varepsilon)$ in $L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$. As for the last term, it tends to 0 in the sense of distributions, for b^ε is bounded in $L^2(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$.

In order to complete the proof of the statement, it is only a matter of establishing that $\mathcal{P}\vec u^\varepsilon$ converges in the sense of distributions to the solution $\vec v$ of (5.2). To achieve it, we project the velocity equation onto divergence-free vector fields:

$$\begin{aligned}\partial_t\mathcal{P}\vec u^\varepsilon-\mu\Delta\mathcal{P}\vec u^\varepsilon=&-\mathcal{P}(\vec u^\varepsilon\cdot\nabla\vec u^\varepsilon)\\&+\bar\varrho^{-1}\mathcal{P}\big(k_2(\widetilde\varepsilon b^\varepsilon)\mathcal{A}\vec u^\varepsilon\big)+(n\bar\varrho)^{-1}(\sigma_a+\sigma_s)\mathcal{P}\big((1+k_4(\widetilde\varepsilon b^\varepsilon))\vec j_1^\varepsilon\big).\end{aligned}\tag{5.6}$$

Because $\mathcal{Q}\vec u=0$, the left-hand side converges to $\partial_t\vec u-\mu\Delta\vec u$. Next, using that b^ε (resp. $\vec u^\varepsilon$) is bounded in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$ (resp. $L^1(\mathbb{R}_+;\dot B^{\frac n2+1}_{2,1})$), we see that the second term in the right-hand side is $\mathcal{O}(\varepsilon)$ in $L^1(\mathbb{R}_+;\dot B^{\frac n2-2}_{2,1})$ (or rather in $L^1(\mathbb{R}_+\times\mathbb{R}^n)$ if $d=2$ owing to endpoint product estimates in Besov spaces). The above considerations also show that the last term is $\mathcal{O}(\varepsilon)$ in the space $(L^1+L^2)(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$.

Let us finally study the convergence of $\mathcal{P}(\vec{u}^\varepsilon\cdot\nabla\vec{u}^\varepsilon)$. We note that
$$\vec{u}^\varepsilon\cdot\nabla\vec{u}^\varepsilon=\frac12\nabla|\mathcal{Q}\vec{u}^\varepsilon|^2+\mathcal{P}\vec{u}^\varepsilon\cdot\nabla\vec{u}^\varepsilon+\mathcal{Q}\vec{u}^\varepsilon\cdot\nabla\mathcal{P}\vec{u}^\varepsilon.$$
Projecting the first term onto divergence free vector fields gives 0, so we just have to study the convergence of $\mathcal{P}(\mathcal{P}\vec{u}^\varepsilon\cdot\nabla\vec{u}^\varepsilon)$ and $\mathcal{P}(\mathcal{Q}\vec{u}^\varepsilon\cdot\nabla\mathcal{P}\vec{u}^\varepsilon)$. Now, a further glance at (5.6) shows that the r.h.s. is bounded in, say, $L^{\frac43}(\mathbb{R}_+;\dot{B}^{\frac n2-\frac32}_{2,1})+L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$.

Indeed, by interpolation, we see that $\nabla\vec{u}^\varepsilon$ is bounded in $L^{\frac43}(\mathbb{R}_+;\dot{B}^{\frac n2-\frac12}_{2,1})$. As $\vec{u}^\varepsilon$ and $\varepsilon b^\varepsilon$ are bounded in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$ and $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2}_{2,1})$, respectively, product laws in Besov spaces ensure that the first two terms of the r.h.s. of (5.6) are bounded in $L^{\frac43}(\mathbb{R}_+;\dot{B}^{\frac n2-\frac32}_{2,1})$. The last term is bounded in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$ because $\varepsilon b^\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2}_{2,1})$ and $\vec{j}_1^\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$. This finally means that $\partial_t\mathcal{P}\vec{u}^\varepsilon$ is bounded in $L^{\frac43}(\mathbb{R}_+;\dot{B}^{\frac n2-\frac32}_{2,1})+L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$, whence $\mathcal{P}\vec{u}^\varepsilon$ is bounded in $\mathcal{C}^{\frac14}_{loc}(\mathbb{R}_+;B^{\frac n2-\frac32}_{2,1,loc})$. As $\mathcal{P}\vec{u}^\varepsilon$ is also bounded in $\mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$, and as the embedding of $B^{\frac n2-1}_{2,1}$ in $B^{\frac n2-\frac32}_{2,1}$ is locally compact (see e.g. [1], page 108), we conclude that, up to extraction, for all $\phi\in\mathcal{S}(\mathbb{R}^d)$ and $T>0$,
$$\phi\mathcal{P}\vec{u}^\varepsilon\longrightarrow\phi\mathcal{P}\vec{u}\quad\text{in}\quad\mathcal{C}([0,T];\dot{B}^{\frac n2-\frac32}_{2,1}).$$
Combining with the uniform bounds in $\mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac n2+1}_{2,1})$, this allows to conclude that $\mathcal{P}(\mathcal{P}\vec{u}^\varepsilon\cdot\nabla\vec{u}^\varepsilon)$ and $\mathcal{P}(\mathcal{Q}\vec{u}^\varepsilon\cdot\nabla\mathcal{P}\vec{u}^\varepsilon)$ converge to $\mathcal{P}(\mathcal{P}\vec{u}\cdot\nabla\vec{u})$ and $\mathcal{P}(\mathcal{Q}\vec{u}\cdot\nabla\mathcal{P}\vec{u})$, respectively, in the sense of distributions. As $\mathcal{P}\vec{u}=\vec{u}$, this completes the proof that $\vec{u}$ satisfies (5.2) for some $\nabla\Pi$. □

5.2. **Strong convergence results.** This part is devoted to the proof of a more precise result, in the whole space case, that does take advantage of the dispersive properties of the acoustic wave equation.

Theorem 5.2. *Assume that the fluid domain is $\mathbb{R}^n$ $(n\geq2)$ and consider a family of data $(b_0^\varepsilon,\vec{u}_0^\varepsilon,j_{0,0}^\varepsilon,\vec{j}_{1,0}^\varepsilon)$ as in Theorem 2.1. Let $(b^\varepsilon,\vec{u}^\varepsilon,j_0^\varepsilon,\vec{j}_1^\varepsilon)$ be the corresponding solution of System (2.1). Then (5.5) is fulfilled by $(j_0^\varepsilon,\vec{j}_1^\varepsilon)$. Furthermore, $(b^\varepsilon,\mathcal{Q}\vec{u}^\varepsilon)\to0$ and $\mathcal{P}\vec{u}^\varepsilon\to\vec{v}$ (with $\vec{v}$ solution to (5.2)) in the following sense :*

- *Case $n\geq4$: $\sqrt{\nu}\|(b^\varepsilon,\mathcal{Q}\vec{u}^\varepsilon)\|_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac np-\frac12}_{p,1})}\leq C\varepsilon^{1/2}I_0^\varepsilon$ for all $p\in[p_c,\infty]$ with $p_c:=(2n-2)/(n-3)$, and*

$$\|\mathcal{P}\vec{u}^\varepsilon-\vec{v}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac np+\frac12}_{p,1}+\dot{B}^{\frac n2+1}_{2,1})\cap L^\infty(\mathbb{R}_+;\dot{B}^{\frac np-\frac32}_{p,1}+\dot{B}^{\frac n2-1}_{2,1})}\leq C_\nu\big(\varepsilon^{1/2}I_0^\varepsilon+\|\mathcal{P}\vec{u}_0^\varepsilon-\vec{v}_0\|_{\dot{B}^{\frac np-\frac32}_{p,1}+\dot{B}^{\frac n2-1}_{2,1}}\big).\tag{5.7}$$

- *Case $n=3$: $\sqrt{\nu}\|(b^\varepsilon,\mathcal{Q}\vec{u}^\varepsilon)\|_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac4q-\frac12}_{q,1})}\leq C\varepsilon^{\frac12-\frac1q}I_0^\varepsilon$ for all $q\in[2,\infty)$, and*

$$\|\mathcal{P}\vec{u}^\varepsilon-\vec{v}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac4q+\frac12}_{q,1}+\dot{B}^{\frac52}_{2,1})\cap L^\infty(\mathbb{R}_+;\dot{B}^{\frac4q-\frac32}_{q,1}+\dot{B}^{\frac12}_{2,1})}\leq C_\nu\big(\varepsilon^{\frac12-\frac1q}I_0^\varepsilon+\|\mathcal{P}\vec{u}_0^\varepsilon-\vec{v}_0\|_{\dot{B}^{\frac4q-\frac32}_{p,1}+\dot{B}^{\frac12}_{2,1}}\big).\tag{5.8}$$

- *Case $n=2$:* $\sqrt{\nu}\|(b^\varepsilon,\mathcal{Q}\vec{u}^\varepsilon)\|_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac14}_{q,1})}\leq C\varepsilon^{\frac14-\frac{1}{2q}}I_0^\varepsilon$ *for all* $q\in[2,6],$ *and*

$$\text{(5.9)}\quad \|\mathcal{P}\vec{u}^\varepsilon-\vec{v}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}+\frac34}_{q,1}+\dot{B}^2_{2,1})\cap L^\infty(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac54}_{q,1}+\dot{B}^0_{2,1})}\leq C_\nu\big(\varepsilon^{\frac14-\frac{1}{2q}}I_0^\varepsilon+\|\mathcal{P}\vec{u}_0^\varepsilon-\vec{v}_0\|_{\dot{B}^{\frac{5}{2q}-\frac54}_{p,1}+\dot{B}^0_{2,1}}\big).$$

Proof: We have already established (5.5) in the previous subsection. In order to prove the strong convergence of $(b^\varepsilon,\mathcal{Q}\vec{u}^\varepsilon)$ to 0, it is convenient to rescale the system : defining $(b,\vec{u},j_0,\vec{j}_1)$ according to (2.8), we see that

$$\text{(5.10)}\qquad \begin{cases}\partial_t b+\operatorname{div}\mathcal{Q}\vec{u}=F\\ \partial_t\mathcal{Q}\vec{u}+\nabla b=\vec{G}+\widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2\mathcal{Q}\big((1+k_4(b))\vec{j}_1\big)\end{cases}$$

where the solution $(b,\vec{u},j_0,\vec{j}_1)$ is given by Theorem 4.1,

$$F:=\mathcal{Q}\big(k_1(b)\operatorname{div}\vec{u}-\vec{u}\cdot\nabla b\big)\quad\text{and}\quad \vec{G}:=\mathcal{Q}\big(-\vec{u}\cdot\nabla\vec{u}+\nu\big((1+k_2(b))\tilde{\mathcal{A}}\vec{u}\big)+k_3(b)\nabla b\big).$$

Note that the terms F and $\vec{G}$ are exactly those that appear when dealing with the standard low Mach number limit for the barotropic Navier-Stokes equations. Because $\vec{u}$ is bounded in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$, $\nabla\vec{u}$, in $L^1(\mathbb{R}_+;\dot{B}^{\frac n2}_{2,1})$, and b, in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2}_{2,1})\cap L^2(\mathbb{R}_+;\dot{B}^{\frac n2}_{2,1})$, it is easy to show that F and G are bounded in $L^1(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$. More precisely, we have (see [5] for more details)

$$\|(F,\vec{G})\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})}\leq C\|(b,\vec{u},j_0,\vec{j}_1)\|^2_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal{C}}}}.$$

Next, according to (4.11),

$$\|k_4(b)\vec{j}_1\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})}\leq C\|(b,\vec{u},j_0,\vec{j}_1)\|^2_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal{C}}}}.$$

As we do not have any control in $L^1(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$ for the low frequencies of the term $\mathcal{Q}\vec{j}_1$, we proceed differently in low and high frequencies. More precisely, projecting System (5.10) on frequencies larger than $\rho_\varepsilon=\rho_0\sqrt{\varepsilon\nu\widetilde{\mathcal{C}}}$ (we use a smooth cut-off of course), and taking advantage of Strichartz estimates for the acoustic wave operator (see e.g. [1], Chap. 10), we get in dimension $n\geq4$, for all $p\in[p_c,+\infty]$,

$$\text{(5.11)}\quad \sqrt{\nu}\|(b,\mathcal{Q}\vec{u})\|^{h,\rho_\varepsilon}_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac np-\frac12}_{p,1})}\leq C\big(\|(b_0,\mathcal{Q}\vec{u}_0)\|^{h,\rho_\varepsilon}_{\dot{B}^{\frac n2-1}_{2,1}} + \|(F,\vec{G})\|^{h,\rho_\varepsilon}_{L^1(\dot{B}^{\frac n2-1}_{2,1})}+\widetilde{\mathcal{C}}^{1/2}\varepsilon^2\|\mathcal{Q}(k_4(b)\vec{j}_1)\|^{h,\rho_\varepsilon}_{L^1(\dot{B}^{\frac n2-1}_{2,1})}+\varepsilon^2\widetilde{\mathcal{C}}^{1/2}\|\mathcal{Q}\vec{j}_1\|^{h,\rho_\varepsilon}_{L^1(\dot{B}^{\frac n2-1}_{2,1})}\big).$$

To handle the low frequencies, we write

$$\partial_t b+\operatorname{div}\mathcal{Q}\vec{u}=F,$$
$$\partial_t\mathcal{Q}\vec{u}+\Big(1+\tfrac{\widetilde{\alpha}\widetilde{\varsigma}^2\varepsilon^2}{\widetilde{\beta}\widetilde{\gamma}}\Big)\nabla b=\vec{G}+\widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2\mathcal{Q}(k_4(b)\vec{j}_1)+\widetilde{\varsigma}\widetilde{\mathcal{C}}^{1/2}\varepsilon^2\mathcal{Q}\vec{j}_1.$$

The left-hand side reduces to the acoustic wave operator with velocity 1. Indeed, it suffices to change b to $\sqrt{1+\frac{\widetilde{\alpha}\widetilde{\varsigma}^2\varepsilon^2}{\widetilde{\beta}\widetilde{\gamma}}}\,b$ and to rescale the time variable by a factor $\sqrt{1+\frac{\widetilde{\alpha}\widetilde{\varsigma}^2\varepsilon^2}{\widetilde{\beta}\widetilde{\gamma}}}$. Thus we get the same Strichartz estimates as for the previous acoustic wave operator, up to some harmless constant tending to the usual one when ε goes to 0. If $n\geq 4$ then we thus get for all $p\in[p_c,+\infty]$,

$$\sqrt{\nu}\|(b,\mathcal{Q}\vec{u})\|^{\ell,\rho_\varepsilon}_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-\frac12}_{p,1})}\leq C\big(\|(b_0,\mathcal{Q}\vec{u}_0)\|^{\ell,\rho_\varepsilon}_{\dot{B}^{\frac n2-1}_{2,1}}$$
$$+\|(F,\vec{G})\|^{\ell,\rho_\varepsilon}_{L^1(\dot{B}^{\frac n2-1}_{2,1})}+\widetilde{\mathcal{C}}^{1/2}\varepsilon^2\|\mathcal{Q}(k_4(b)\vec{j}_1)\|^{\ell,\rho_\varepsilon}_{L^1(\dot{B}^{\frac n2-1}_{2,1})}+\widetilde{\mathcal{C}}^{1/2}\varepsilon^2\|\mathcal{Q}\vec{j}_1\|^{\ell,\rho_\varepsilon}_{L^1(\dot{B}^{\frac n2-1}_{2,1})}\big).$$

Combining that inequality with (5.11), we thus get

$$\sqrt{\nu}\|(b,\mathcal{Q}\vec{u})\|_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-\frac12}_{p,1})}\leq C\Big(\|(b_0,\mathcal{Q}\vec{u}_0)\|_{\dot{B}^{\frac n2-1}_{2,1}}$$
$$+\|(b,\vec{u},j_0,\vec{j}_1)\|^2_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal{C}}}}+\frac{\varepsilon}{\widetilde{\mathcal{C}}^{1/2}\nu}\|(b,\vec{u},j_0,\vec{j}_1)\|_{E^{\frac n2-1}_{\varepsilon,\nu,\widetilde{\mathcal{C}}}}\Big).$$

Setting $I_0:=\|\vec{u}_0,j_{0,0},\vec{j}_{1,0}\|_{\dot{B}^{\frac n2-1}_{2,1}}+\|b_0\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac n2-1}_{2,1}}+\nu\|b_0\|^{h,\nu^{-1}}_{\dot{B}^{\frac n2}_{2,1}}$ and using the estimates of Theorem 4.1, we conclude that

$$\sqrt{\nu}\|(b,\mathcal{Q}\vec{u})\|_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-\frac12}_{p,1})}\leq CI_0\quad\text{for all}\ \ p\in[p_c,\infty].\tag{5.12}$$

Resuming to the original variables gives the announced rate of convergence for $(b^\varepsilon,\mathcal{Q}\vec{u}^\varepsilon)$ if $n\geq 4$.

In the case $n=2,3$ the above arguments have to be slightly modified as less Strichartz inequalities are available. At the same time, the bounds for F, $\vec{G}$ $\mathcal{Q}\vec{j}_1$ and so on, in $L^1(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$ remain unchanged.

More precisely, if $n=3$ then we get instead of (5.12):

$$\nu^{\frac12-\frac1p}\|(b,\mathcal{Q}\vec{u})\|_{\widetilde{L}^{\frac{2p}{p-2}}(\mathbb{R}_+;\dot{B}^{\frac2p-\frac12}_{p,1})}\leq CI_0\quad\text{for all}\ \ p\in[2,\infty).\tag{5.13}$$

Recall that the bounds of Theorem 4.1 imply that

$$\nu\|(b,\mathcal{Q}\vec{u})\|^{\ell,\nu^{-1}}_{L^1(\mathbb{R}_+;\dot{B}^{\frac52}_{2,1})}\leq CI_0.\tag{5.14}$$

From interpolation, we discover that

$$\|(b,\mathcal{Q}\vec{u})\|^{\ell,\nu^{-1}}_{L^2(\mathbb{R}_+;\dot{B}^{\frac4q-\frac12}_{q,1})}\lesssim\big(\|(b,\mathcal{Q}\vec{u})\|^{\ell,\nu^{-1}}_{L^1(\mathbb{R}_+;\dot{B}^{\frac52}_{2,1})}\big)^{\frac{2}{p+2}}\big(\|(b,\mathcal{Q}\vec{u})\|^{\ell,\nu^{-1}}_{L^{\frac{2p}{p-2}}(\mathbb{R}_+;\dot{B}^{\frac2p-\frac12}_{p,1})}\big)^{\frac{p}{p+2}}$$

with $q:=1+p/2$.

Hence putting (5.13) and (5.14) together gives

$$\sqrt{\nu}\|(b,\mathcal{Q}\vec{u})\|^{\ell,\nu^{-1}}_{L^2(\mathbb{R}_+;\dot{B}^{\frac4q-\frac12}_{q,1})}\leq CI_0\quad\text{for all}\ \ q\in[2,\infty).\tag{5.15}$$

To handle high frequencies, we just have to notice that, because $3/q\geq 4/q-1/2$ for $q\geq 2$, we have the following chain of inequalities:

$$\|(b,\mathcal{Q}\vec{u})\|^{h,\nu^{-1}}_{\dot{B}^{\frac4q-\frac12}_{q,1}}\lesssim\nu^{\frac12-\frac1q}\|(b,\mathcal{Q}\vec{u})\|^{h,\nu^{-1}}_{\dot{B}^{\frac3q}_{q,1}}\lesssim\nu^{\frac12-\frac1q}\|(b,\mathcal{Q}\vec{u})\|^{h,\nu^{-1}}_{\dot{B}^{\frac32}_{2,1}}.\tag{5.16}$$

Remember that Theorem 4.1 yields

$$\sqrt{\nu}\|(b,\mathcal{Q}\vec{u})\|^{h,\nu^{-1}}_{L^2(\mathbb{R}_+;\dot{B}^{\frac32}_{2,1})}\leq CI_0$$

which, together with (5.15) and (5.16) implies, if $\nu \leq \nu_0$, that

$$\sqrt{\nu}\|(b,\mathcal{Q}\vec{u})\|_{L^2(\mathbb{R}_+;\dot{B}^{\frac{4}{q}-\frac{1}{2}}_{q,1})} \leq CI_0 \quad \text{for all} \quad q\in[2,\infty),$$

whence the desired result of convergence for $(b^\varepsilon,\mathcal{Q}\vec{u}^\varepsilon)$, after rescaling.

Let us finally bound $(b,\mathcal{Q}\vec{u})$ in the case $n=2$. Then Strichartz estimates combined with the above bounds in $L^1(\mathbb{R}_+;\dot{B}^0_{2,1})$ imply that for all $p\in[2,+\infty]$,

$$\|(b,\mathcal{Q}\vec{u})\|_{\widetilde{L}^{\frac{4p}{p-2}}(\mathbb{R}_+;\dot{B}^{\frac{3}{2p}-\frac{3}{4}}_{p,1})} + \|(b,\mathcal{Q}\vec{u})\|^{\ell,\nu^{-1}}_{L^1(\mathbb{R}_+;\dot{B}^2_{2,1})} \leq CI_0. \tag{5.17}$$

From interpolation, we discover that

$$\|(b,\mathcal{Q}\vec{u})\|^{\ell,\nu^{-1}}_{L^2(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{1}{4}}_{q,1})} \lesssim \big(\|(b,\mathcal{Q}\vec{u})\|^{\ell,\nu^{-1}}_{L^1(\mathbb{R}_+;\dot{B}^2_{2,1})}\big)^{\frac{p+2}{3p+2}}\big(\|(b,\mathcal{Q}\vec{u})\|^{\ell,\nu^{-1}}_{L^{\frac{4p}{p-2}}(\mathbb{R}_+;\dot{B}^{\frac{3}{2p}-\frac{3}{4}}_{p,1})}\big)^{\frac{2p}{3p+2}}$$

with $q:=(6p+4)/(p+6)\in[2,6]$.

Therefore (5.17) implies that

$$\sqrt{\nu}\|(b,\mathcal{Q}\vec{u})\|^{h,\nu^{-1}}_{L^2(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{1}{4}}_{q,1})} \leq CI_0 \quad \text{for all} \quad q\in[2,6]. \tag{5.18}$$

Next, because $q\geq 2$, we have

$$\|(b,\mathcal{Q}\vec{u})\|^{h,\nu^{-1}}_{\dot{B}^{\frac{5}{2q}-\frac{1}{4}}_{q,1}} \lesssim \nu^{\frac{1}{4}-\frac{1}{2q}}\|(b,\mathcal{Q}\vec{u})\|^{h,\nu^{-1}}_{\dot{B}^{\frac{2}{q}}_{q,1}} \lesssim \nu^{\frac{1}{4}-\frac{1}{2q}}\|(b,\mathcal{Q}\vec{u})\|^{h,\nu^{-1}}_{\dot{B}^1_{2,1}}.$$

Then combining with the bounds of Theorem 4.1 eventually leads to

$$\sqrt{\nu}\|(b,\mathcal{Q}\vec{u})\|_{L^2(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{1}{4}}_{q,1})} \leq CI_0 \quad \text{for all} \quad q\in[2,6],$$

whence the desired result of convergence for $(b^\varepsilon,\mathcal{Q}\vec{u}^\varepsilon)$, after rescaling.

Let us finally go to the proof of the convergence of $\mathcal{P}\vec{u}^\varepsilon$. Subtracting (5.2) from the velocity equation (projected onto divergence free vector fields) of (2.1), and setting $\delta\vec{v}^\varepsilon := \mathcal{P}\vec{u}^\varepsilon - v$, we get

$$\begin{aligned}\partial_t\delta\vec{v}^\varepsilon - \frac{\mu}{\bar\varrho}\Delta\delta\vec{v}^\varepsilon = -\mathcal{P}(\mathcal{P}\vec{u}^\varepsilon\cdot\nabla\delta\vec{v}^\varepsilon + \delta\vec{v}^\varepsilon\cdot\nabla\vec{v} + \vec{u}^\varepsilon\cdot\nabla\mathcal{Q}\vec{u}^\varepsilon + \mathcal{Q}\vec{u}^\varepsilon\cdot\nabla\mathcal{P}\vec{u}^\varepsilon)\\ + \frac{1}{\bar\varrho}\mathcal{P}\big(k_2(\widetilde{\varepsilon}b^\varepsilon)\mathcal{A}\vec{u}^\varepsilon\big) + \left(\frac{\sigma_a+\sigma_s}{n\bar\varrho}\right)\mathcal{P}\big((1+k_4(\widetilde{\varepsilon}b^\varepsilon))\vec{j}^\varepsilon_1\big).\end{aligned} \tag{5.19}$$

Except for the last term, the proof is exactly the same as for the barotropic case (see [1, 5]). More precisely, the first two terms of the r.h.s. satisfy *linear* estimates (with small coefficient) with respect to $\delta\vec{v}^\varepsilon$, while the next three terms are expected to be of order ε^α for some $\alpha>0$, owing to the convergence of $(b^\varepsilon,\mathcal{Q}\vec{u}^\varepsilon)$ and uniform estimates. Let us give more details. To simplify the presentation, we do not keep track of the dependency of the estimates, on ν.

Let us first consider the case $n\geq 4$. Then we have for all $p\in[p_c,+\infty]$

$$\begin{aligned}\|\vec{u}^\varepsilon\cdot\nabla\mathcal{Q}\vec{u}^\varepsilon + \mathcal{Q}\vec{u}^\varepsilon\cdot\nabla\mathcal{P}\vec{u}^\varepsilon\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-\frac{3}{2}}_{p,1})} &\lesssim \|\vec{u}^\varepsilon\|_{L^2(\mathbb{R}_+;\dot{B}^{\frac{n}{2}}_{2,1})}\|\mathcal{Q}\vec{u}^\varepsilon\|_{L^2(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1})}\\ &\lesssim \varepsilon^{1/2}(I_0^\varepsilon)^2\end{aligned}$$

and, because

$$\|b^\varepsilon\|_{\dot B^{\frac n2-\alpha}_{2,1}} \lesssim \varepsilon^{\alpha-1}\big(\|b^\varepsilon\|^{\ell,\frac{1}{\varepsilon\nu}}_{\dot B^{\frac n2-1}_{2,1}}+\varepsilon\|b^\varepsilon\|^{h,\frac{1}{\varepsilon\nu}}_{\dot B^{\frac n2}_{2,1}}\big)\quad\text{for all}\quad \alpha\in[0,1], \tag{5.20}$$

we have

$$\begin{aligned}\|k_2(\widetilde\varepsilon b^\varepsilon)\mathcal{A}\vec u^\varepsilon\|_{L^1(\mathbb{R}_+;\dot B^{\frac np-\frac32}_{p,1})} &\lesssim \|k_2(\widetilde\varepsilon b^\varepsilon)\mathcal{A}\vec u^\varepsilon\|_{L^1(\mathbb{R}_+;\dot B^{\frac n2-\frac32}_{2,1})}\\ &\lesssim \|\widetilde\varepsilon b^\varepsilon\|_{L^\infty(\mathbb{R}_+;\dot B^{\frac n2-\frac12}_{2,1})}\|\nabla^2\vec u^\varepsilon\|_{L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})}\\ &\lesssim \varepsilon^{1/2}(I_0^\varepsilon)^2.\end{aligned}$$

Keeping in mind the regularity estimates for the heat equation, it seems thus natural to bound $\delta\vec v^\varepsilon$ in $L^1(\mathbb{R}_+;\dot B^{\frac np+\frac12}_{p,1})\cap L^\infty(\mathbb{R}_+;\dot B^{\frac np-\frac12}_{p,1})$. In contrast with the nonradiative situation however, this is not possible owing to the term $\mathcal{P}\vec j_1^\varepsilon$ which is only damped, and not dispersed. More precisely, we have

$$\mathcal{P}\big((1+k_4(\widetilde\varepsilon b^\varepsilon))\vec j_1^\varepsilon\big)=\mathcal{P}\vec j_1^\varepsilon+\mathcal{P}(k_4(\widetilde\varepsilon b^\varepsilon)\vec j_1^\varepsilon).$$

By following the argument leading to (4.11) and using the estimates supplied by Theorem 2.1, it is easy to bound the last term as follows:

$$\|\mathcal{P}(k_4(\widetilde\varepsilon b^\varepsilon)\vec j_1^\varepsilon)\|_{L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})}\le C\widetilde{\mathcal{C}}^{-1/2}\varepsilon I_0^\varepsilon. \tag{5.21}$$

For the first term, we just have to notice that $\mathcal{P}\vec j_1^\varepsilon=\mathcal{P}\vec{\mathrm{j}}_1^\varepsilon$, hence (5.21) is also fulfilled.

So finally the r.h.s. of (5.19) has to be bounded in $L^1(\mathbb{R}_+;\dot B^{\frac np-\frac32}_{p,1}+\dot B^{\frac n2-1}_{2,1})$ and thus $\delta\vec v^\varepsilon$, in $L^\infty(\mathbb{R}_+;\dot B^{\frac np-\frac32}_{p,1}+\dot B^{\frac n2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac np+\frac12}_{p,1}+\dot B^{\frac n2+1}_{2,1})$ (if $\delta\vec v_0^\varepsilon$ satisfies suitable assumptions of course). Now, because

$$\begin{aligned}\|\mathcal{P}\vec u^\varepsilon\cdot\nabla\delta\vec v^\varepsilon+\delta\vec v^\varepsilon\cdot\nabla\vec v\|_{L^1(\mathbb{R}_+;\dot B^{\frac np-\frac32}_{p,1}+\dot B^{\frac n2-1}_{2,1})} &\lesssim \|\mathcal{P}\vec u^\varepsilon,\vec v\|_{L^2(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})}\|\delta\vec v^\varepsilon\|_{L^2(\mathbb{R}_+;\dot B^{\frac np-\frac12}_{p,1}+\dot B^{\frac n2}_{2,1})}\\ &\le CI_0^\varepsilon\|\delta\vec v^\varepsilon\|_{L^2(\mathbb{R}_+;\dot B^{\frac np-\frac12}_{p,1}+\dot B^{\frac n2}_{2,1})}\end{aligned}$$

it is easy to conclude to (5.7), if I_0^ε is small enough.

In the three-dimensional case, we claim that

$$\delta\vec v^\varepsilon\to0\quad\text{in}\quad L^\infty(\mathbb{R}_+;\dot B^{\frac4q-\frac32}_{q,1}+\dot B^{\frac12}_{2,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac4q-\frac32}_{q,1}+\dot B^{\frac12}_{2,1}),\quad\text{for all}\quad q\in[2,\infty).$$

To achieve this result, it suffices to prove suitable estimates for the r.h.s. of (5.19) in $L^1(\mathbb{R}_+;\dot B^{\frac4q-\frac32}_{q,1}+\dot B^{\frac12}_{2,1})$. We have

$$\begin{aligned}\|\vec u^\varepsilon\cdot\nabla\mathcal{Q}\vec u^\varepsilon+\mathcal{Q}\vec u^\varepsilon\cdot\nabla\mathcal{P}\vec u^\varepsilon\|_{L^1(\mathbb{R}_+;\dot B^{\frac4q-\frac32}_{q,1})} &\lesssim \|\vec u^\varepsilon\|_{L^2(\mathbb{R}_+;\dot B^{\frac32}_{2,1})}\|\mathcal{Q}\vec u^\varepsilon\|_{L^2(\mathbb{R}_+;\dot B^{\frac4q-\frac12}_{q,1})}\\ &\lesssim \varepsilon^{\frac12-\frac1q}I_0^\varepsilon\end{aligned}$$

$$\begin{aligned}\|\mathcal{P}\vec u^\varepsilon\cdot\nabla\delta\vec v^\varepsilon+\delta\vec v^\varepsilon\cdot\nabla\vec v\|_{L^1(\mathbb{R}_+;\dot B^{\frac4q-\frac32}_{q,1}+\dot B^{\frac12}_{2,1})} &\lesssim \|\mathcal{P}\vec u^\varepsilon,\vec v\|_{L^2(\mathbb{R}_+;\dot B^{\frac32}_{2,1})}\|\delta\vec v^\varepsilon\|_{L^2(\mathbb{R}_+;\dot B^{\frac4q-\frac12}_{q,1}+\dot B^{\frac32}_{2,1})}\\ &\le CI_0^\varepsilon\|\delta\vec v^\varepsilon\|_{L^2(\mathbb{R}_+;\dot B^{\frac4q-\frac12}_{q,1}+\dot B^{\frac32}_{2,1})}\end{aligned}$$

and, thanks to (5.20) with $\alpha = 1/2 - 1/q$,

$$\begin{aligned}\|k_2(\widetilde{\varepsilon b}^\varepsilon)\mathcal{A}\vec{u}^\varepsilon\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{4}{q}-\frac{3}{2}}_{q,1})} &\lesssim \|k_2(\widetilde{\varepsilon b}^\varepsilon)\mathcal{A}\vec{u}^\varepsilon\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{1}{q}}_{2,1})}\\ &\lesssim \|\widetilde{\varepsilon b}^\varepsilon\|_{L^\infty(\mathbb{R}_+;\dot{B}^{1+\frac{1}{q}}_{2,1})}\|\nabla^2\vec{u}^\varepsilon\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{1}{2}}_{2,1})}\\ &\lesssim \varepsilon^{\frac12-\frac1q}I_0^\varepsilon.\end{aligned}$$

Keeping (5.21) in mind, we can thus conclude to (5.8).

In the two-dimensional case, we just have to prove suitable estimates for the r.h.s. of (5.19) in $L^1(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{5}{4}}_{q,1}+\dot{B}^0_{2,1})$. We have

$$\begin{aligned}\|\vec{u}^\varepsilon\cdot\nabla\mathcal{Q}\vec{u}^\varepsilon+\mathcal{Q}\vec{u}^\varepsilon\cdot\nabla\mathcal{P}\vec{u}^\varepsilon\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{5}{4}}_{q,1})} &\lesssim \|\vec{u}^\varepsilon\|_{L^2(\mathbb{R}_+;\dot{B}^1_{2,1})}\|\mathcal{Q}\vec{u}^\varepsilon\|_{L^2(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{1}{4}}_{q,1})}\\ &\lesssim \varepsilon^{\frac14-\frac{1}{2q}}I_0^\varepsilon\\ \|\mathcal{P}\vec{u}^\varepsilon\cdot\nabla\delta\vec{v}^\varepsilon+\delta\vec{v}^\varepsilon\cdot\nabla\vec{v}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{5}{4}}_{q,1}+\dot{B}^{\frac12}_{2,1})} &\lesssim \|\mathcal{P}\vec{u}^\varepsilon,\vec{v}\|_{L^2(\mathbb{R}_+;\dot{B}^1_{2,1})}\|\delta\vec{v}^\varepsilon\|_{L^2(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{1}{4}}_{q,1}+\dot{B}^1_{2,1})}\\ &\le CI_0^\varepsilon\|\delta\vec{v}^\varepsilon\|_{L^2(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{1}{4}}_{q,1}+\dot{B}^1_{2,1})}\end{aligned}$$

and, thanks to (5.20) with $\alpha = 1/4 - 1/(2q)$,

$$\begin{aligned}\|k_2(\widetilde{\varepsilon b}^\varepsilon)\mathcal{A}\vec{u}^\varepsilon\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{5}{2q}-\frac{5}{4}}_{q,1})} &\lesssim \|k_2(\widetilde{\varepsilon b}^\varepsilon)\mathcal{A}\vec{u}^\varepsilon\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{1}{2q}-\frac14}_{2,1})}\\ &\lesssim \|\widetilde{\varepsilon b}^\varepsilon\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\frac34+\frac{1}{2q}}_{2,1})}\|\nabla^2\vec{u}^\varepsilon\|_{L^1(\mathbb{R}_+;\dot{B}^0_{2,1})}\\ &\lesssim \varepsilon^{\frac14-\frac{1}{2q}}I_0^\varepsilon.\end{aligned}$$

Keeping (5.21) in mind, we can thus conclude to (5.8). □

Appendix A

We here provide exponential decay estimates for the following linear system of ordinary differential equations with X and Y complex-valued:

$$\left\{\begin{array}{l}\partial_t X + a\rho Y - b\rho^2 X = A\\ \partial_t Y - c\rho X + d\rho^2 Y = B.\end{array}\right. \tag{A.1}$$

Above, ρ stands for a given nonnegative small parameter and a, b, c and d are four real numbers satisfying the stability condition

$$a>0,\quad c>0\quad\text{and}\quad d-b>0. \tag{A.2}$$

Even though (A.1) may be solved explicitly, thus giving the desired (and optimal) decay results, we here aim at recovering such results by means of an energy type method. We start with the following identities:

$$\begin{aligned}&\frac12\frac{d}{dt}\big(c|X|^2+a|Y|^2\big)-bc\rho^2|X|^2+ad\rho^2|Y|^2=\mathrm{Re}\,(cA\bar{X}+aB\bar{Y}),\\ &\frac{d}{dt}\mathrm{Re}\,(X\bar{Y})+a\rho|Y|^2-c\rho|X|^2+(d-b)\rho^2\mathrm{Re}\,(X\bar{Y})=\mathrm{Re}\,(B\bar{X}+A\bar{Y})\end{aligned}$$

from which we easily get for any real number η,

$$\begin{aligned}&\frac12\frac{d}{dt}\big(c|X|^2+a|Y|^2-2\rho\eta\mathrm{Re}\,(X\bar{Y})\big)+(\eta-b)c\rho^2|X|^2\\ &\quad+(d-\eta)a\rho^2|Y|^2+\eta(b-d)\rho^3\mathrm{Re}\,(X\bar{Y})=\mathrm{Re}\,\big(cA\bar{X}+aB\bar{Y}-2\rho\eta(B\bar{X}+A\bar{Y})\big).\end{aligned} \tag{A.3}$$

Choosing η so that $d-\eta=\eta-b$, that is to say $\eta:=\frac{b+d}{2}$, we discover that the *Lyapunov functional* $\mathcal{L}_\rho^2:=c|X|^2+a|Y|^2-\rho(d+b)\mathrm{Re}\,(X\bar{Y})$ satisfies

$$\text{(A.4)}\quad \frac{1}{2}\frac{d}{dt}\mathcal{L}_\rho^2+\left(\frac{d-b}{2}\right)\rho^2\big(c|X|^2+a|Y|^2\big)+\left(\frac{b^2-d^2}{2}\right)\rho^3\mathrm{Re}\,(X\bar{Y})$$
$$=\mathrm{Re}\,\big(cA\bar{X}+aB\bar{Y}-\rho(b+d)(B\bar{X}+A\bar{Y})\big).$$

Now, from the observation that

$$|\mathrm{Re}\,(X\bar{Y})|\leq\frac{1}{2\sqrt{ac}}\big(c|X|^2+a|Y|^2\big),$$

we gather that, whenever $\rho\leq\frac{\sqrt{ac}}{|b+d|}$, we have

$$\left|\left(\frac{b^2-d^2}{2}\right)\rho^3\mathrm{Re}\,(X\bar{Y})\right|\leq\left(\frac{d-b}{4}\right)\rho^2\big(c|X|^2+a|Y|^2\big)$$

and

$$\text{(A.5)}\quad \frac{1}{2}\big(c|X|^2+a|Y|^2\big)\leq\mathcal{L}_\rho^2\leq\frac{3}{2}\big(c|X|^2+a|Y|^2\big).$$

If $A\equiv B\equiv 0$ then resuming to (A.4) leads to

$$\text{(A.6)}\quad \frac{d}{dt}\mathcal{L}_\rho^2+\left(\frac{d-b}{3}\right)\mathcal{L}_\rho^2\leq 0$$

and thus, for any $t\geq 0$,

$$\text{(A.7)}\quad \mathcal{L}_\rho(t)\leq e^{-(\frac{d-b}{6})\rho^2t}\mathcal{L}_\rho(0),$$

which yields, according to (A.5),

$$\text{(A.8)}\quad \sqrt{c|X(t)|^2+a|Y(t)|^2}\leq\sqrt{3}\sqrt{c|X(0)|^2+a|Y(0)|^2}\,e^{-\left(\frac{d-b}{6}\right)\rho^2t}\quad\text{for}\quad \rho\leq\frac{\sqrt{ac}}{|b+d|}\cdot$$

For nonzero source terms, Inequality (A.7) leads through Duhamel formula to

$$\text{(A.9)}\quad \sqrt{c|X(t)|^2+a|Y(t)|^2}\leq\sqrt{3}\,e^{-(\frac{d-b}{6})\rho^2t}\bigg(\sqrt{c|X(0)|^2+a|Y(0)|^2}$$
$$+\int_0^t e^{(\frac{d-b}{6})\tau}\sqrt{c|A|^2+a|B|^2}\,d\tau\bigg)\cdot$$

References

[1] H. Bahouri, J.-Y. Chemin and R. Danchin: *Fourier Analysis and Nonlinear Partial Differential Equations,* Grundlehren der mathematischen Wissenschaften, **343**, Springer (2011).

[2] X. Blanc and B. Després: *Numerical methods for inertial confinement fusion,* Lecture Notes of CEMRACS-10 Summer School: `smai.emath.fr/cemracs/cemracs10/fr_courses.html.`

[3] C. Buet and B. Després, Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics, *J. Quant. Spectroscopy Rad. Transf.*, **85**:385–480, 2004.

[4] S. Chandrasekhar, *Radiative transfer.* Dover Publications, Inc., New York, 1960.

[5] R. Danchin: Zero Mach number limit in critical spaces for compressible Navier-Stokes equations, *Ann. Sci. École Norm. Sup.*, **35**(1), pages 27–75 (2002).

[6] R. Danchin: On the uniqueness in critical spaces for compressible Navier-Stokes equations, *NoDEA Nonlinear Differential Equations Appl.*, **12**(1), 111–128 (2005).

[7] R. Danchin and B. Ducomet: On a simplified model for radiating flows, *Journal of Evolution Equations*, **14**:155–195, 2014.

[8] R. Danchin and B. Ducomet: Diffusive limits for a barotropic model of radiative flow, in progress.

[9] B. Dubroca, M. Seaïd, J.-L. Feugeas, A consistent approach for the coupling of radiation and hydrodynamics at low Mach number, J. of Comput. Phys. 225 (2007) 1039–1065.

[10] B. Ducomet, E. Feireisl, Š. Nečasova: On a model of radiation hydrodynamics, *Ann. I. H. Poincaré, Analyse non linéaire*, **28**:797–812, 2011.

[11] B. Ducomet et Š. Nečasová: Low Mach number limit for a model of radiation hydrodynamics, *Journal of Evolution Equations,* **14**, pages 357–385 (2014).

[12] E. Feireisl and A. Novotný: *Singular limits in thermodynamics of viscous fluids.* Birkhaüser, Basel, 2009.

[13] P.-L. Lions and N. Masmoudi: Une approche locale de la limite incompressible, *C. R. Acad. Sci. Paris Sér. I Math.*, **329**(5):387–392, 1999.

[14] R. B. Lowrie, J. E. Morel, J. A. Hittinger: The coupling of radiation and hydrodynamics, *The Astrophysical Journal*, **521**:432–450, 1999.

[15] B. Mihalas and B. Weibel-Mihalas: *Foundations of radiation hydrodynamics.* Dover Publications, Dover, 1984.

[16] G. C. Pomraning: *Radiation hydrodynamics.* Dover Publications, New York, 2005.

[17] I. Teleaga, M. Seaïd, Simplified radiative models for low Mach number reactive flows, Appl. Math. Modeling 32 (2008) 971–991.

[18] I. Teleaga, M. Seaïd, I. Gasser, A. Klar, J. Struckmeier, Radiation models for thermal flows at low Mach number, J. of Comput. Phys. 215 (2006) 506–525.